



# LaplacesDemon Examples

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## Abstract

The **LaplacesDemon** package in R enables Bayesian inference with any Bayesian model, provided the user specifies the likelihood. This vignette is a compendium of examples of how to specify different model forms.

*Keywords:* ~Bayesian, Bayesian Inference, Laplace's Demon, LaplacesDemon, R, STATISTICAT.

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**LaplacesDemon** (Hall 2011), usually referred to as Laplace's Demon, is an R package that is available on CRAN (R Development Core Team 2011). A formal introduction to Laplace's Demon is provided in an accompanying vignette entitled "**LaplacesDemon** Tutorial", and an introduction to Bayesian inference is provided in the "Bayesian Inference" vignette.

The purpose of this document is to provide users of the **LaplacesDemon** package with examples of a variety of Bayesian methods. It is also a testament to the diverse applicability of **LaplacesDemon** to Bayesian inference.

To conserve space, the examples are not worked out in detail, and only the minimum of necessary materials is provided for using the various methodologies. Necessary materials include the form expressed in notation, data (which is often simulated), initial values, and the `Model` function. The provided data, initial values, and model specification may be copy/pasted into an R file and updated with the `LaplacesDemon` or (usually) `LaplaceApproximation` functions. Although many of these examples update quickly, some examples are computationally intensive.

Notation in this vignette follows these standards: Greek letters represent parameters, lower case letters represent indices, lower case bold face letters represent scalars or vectors, probability distributions are represented with calligraphic font, upper case letters represent index limits, and upper case bold face letters represent matrices.

This vignette will grow over time as examples of more methods become included. Contributed examples are welcome. Please send contributed examples or discovered errors in a similar format in an email to [laplacesdemon@statisticat.com](mailto:laplacesdemon@statisticat.com) for review and testing. All accepted contributions are, of course, credited.

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## 1. ANCOVA

This example is essentially the same as the two-way ANOVA (see section 3), except that a covariate  $\mathbf{X}_{,3}$  has been added, and its parameter is  $\delta$ .

### 1.1. Form

$$\begin{aligned}
 \mathbf{y}_i &\sim \mathcal{N}(\mu_i, \sigma_1^2) \\
 \mu_i &= \alpha + \beta[\mathbf{X}_{i,1}] + \gamma[\mathbf{X}_{i,2}] + \delta\mathbf{X}_{i,2}, \quad i = 1, \dots, N \\
 \epsilon_i &= \mathbf{y}_i - \mu_i \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \beta_j &\sim \mathcal{N}(0, \sigma_2^2), \quad j = 1, \dots, (J-1) \\
 \beta_J &= -\sum_{j=1}^{J-1} \beta_j \\
 \gamma_k &\sim \mathcal{N}(0, \sigma_3^2), \quad k = 1, \dots, (K-1) \\
 \gamma_K &= -\sum_{k=1}^{K-1} \gamma_k
 \end{aligned}$$

$$\delta \sim \mathcal{N}(0, 1000)$$

$$\sigma_m \sim \mathcal{HC}(25), \quad m = 1, \dots, 3$$

## 1.2. Data

```

N <- 100
J <- 5 #Number of levels in factor (treatment) 1
K <- 3 #Number of levels in factor (treatment) 2
X <- matrix(cbind(round(runif(N,0.5,J+0.49)),round(runif(N,0.5,K+0.49)),
  runif(N,-2,2)), N, 3)
alpha <- runif(1,-1,1)
beta <- runif(J,-2,2)
beta[J] <- -sum(beta[1:(J-1)])
gamma <- runif(K,-2,2)
gamma[J] <- -sum(gamma[1:(K-1)])
delta <- runif(1,-2,2)
y <- alpha + beta[X[,1]] + gamma[X[,2]] + delta*X[,3] + rnorm(N,0,0.1)
mon.names <- c("LP","beta[5]","gamma[3]","sigma[1]","sigma[2]","sigma[3]","
  "s.beta","s.gamma","s.epsilon")
parm.names <- parm.names(list(alpha=0, beta=rep(0,J-1), gamma=rep(0,K-1),
  delta=0, log.sigma=rep(0,3)))
MyData <- list(J=J, K=K, N=N, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

## 1.3. Initial Values

```
Initial.Values <- c(0, rep(0,(J-1)), rep(0,(K-1)), 0, rep(log(1),3))
```

## 1.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]
  beta <- rep(NA,Data$J)
  beta[1:(Data$J-1)] <- parm[2:Data$J]
  beta[J] <- -sum(beta[1:(Data$J-1)]) #Sum-to-zero constraint
  gamma <- rep(NA,Data$K)
  gamma[1:(Data$K-1)] <- parm[grep("gamma", Data$parm.names)]
  gamma[K] <- -sum(gamma[1:(Data$K-1)]) #Sum-to-zero constraint
  delta <- parm[grep("delta", Data$parm.names)]
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, sigma[2], log=TRUE))
  gamma.prior <- sum(dnorm(gamma, 0, sigma[3], log=TRUE))
}

```

```

delta.prior <- dnorm(delta, 0, sqrt(1000), log=TRUE)
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
mu <- alpha + beta[Data$X[,1]] + gamma[Data$X[,2]] +
  delta*Data$X[,3]
LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
### Variance Components
s.beta <- sd(beta)
s.gamma <- sd(gamma)
s.epsilon <- sd(Data$y - mu)
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + gamma.prior + delta.prior +
  sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, beta[Data$J],
  gamma[Data$K], sigma, s.beta, s.gamma, s.epsilon), yhat=mu,
  parm=parm)
return(Modelout)
}

```

## 2. ANOVA, One-Way

When  $J = 2$ , this is a Bayesian form of a t-test.

### 2.1. Form

$$\begin{aligned}
 \mathbf{y} &\sim \mathcal{N}(\mu, \sigma_1^2) \\
 \mu_i &= \alpha + \beta[\mathbf{x}_i], \quad i = 1, \dots, N \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \beta_j &\sim \mathcal{N}(0, \sigma_2^2), \quad j = 1, \dots, (J-1) \\
 \beta_J &= -\sum_{j=1}^{J-1} \beta_j \\
 \sigma_{1:2} &\sim \mathcal{HC}(25)
 \end{aligned}$$

### 2.2. Data

```

N <- 100
J <- 5
x <- round(runif(N, 0.5, J+0.49))
alpha <- runif(1,-1,1)
beta <- runif(J,-2,2)
beta[J] <- -sum(beta[1:(J-1)])

```

```

y <- rep(NA, N)
for (i in 1:N) {y[i] <- alpha + beta[x[i]] + rnorm(1,0,0.2)}
mon.names <- c("LP","beta[1]","sigma[1]","sigma[2]")
parm.names <- parm.names(list(alpha=0, beta=rep(0,J-1),
  log.sigma=rep(0,2)))
MyData <- list(J=J, N=N, mon.names=mon.names, parm.names=parm.names, x=x,
  y=y)

```

## 2.3. Initial Values

```
Initial.Values <- c(0, rep(0,(J-1)), rep(log(1),2))
```

## 2.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]
  beta <- rep(NA,Data$J)
  beta[1:(Data$J-1)] <- parm[2:Data$J]
  beta[J] <- -sum(beta[1:(Data$J-1)]) #Sum-to-zero constraint
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, sigma[2], log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  mu <- alpha + beta[Data$x]
  LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,beta[Data$J],
    sigma), yhat=mu, parm=parm)
  return(Modelout)
}

```

## 3. ANOVA, Two-Way

In this representation,  $\sigma^m$  are the superpopulation variance components, **s.beta** and **s.gamma** are the finite-population within-variance components of the factors or treatments, and **s.epsilon** is the finite-population between-variance component.

### 3.1. Form

$$\begin{aligned}
\mathbf{y}_i &\sim \mathcal{N}(\mu_i, \sigma_1^2) \\
\mu_i &= \alpha + \beta[\mathbf{X}_{i,1}] + \gamma[\mathbf{X}_{i,2}], \quad i = 1, \dots, N \\
\epsilon_i &= \mathbf{y}_i - \mu_i \\
\alpha &\sim \mathcal{N}(0, 1000) \\
\beta_j &\sim \mathcal{N}(0, \sigma_2^2), \quad j = 1, \dots, (J-1) \\
\beta_J &= -\sum_{j=1}^{J-1} \beta_j \\
\gamma_k &\sim \mathcal{N}(0, \sigma_3^2), \quad k = 1, \dots, (K-1) \\
\gamma_K &= -\sum_{k=1}^{K-1} \gamma_k \\
\sigma_m &\sim \mathcal{HC}(25), \quad m = 1, \dots, 3
\end{aligned}$$

### 3.2. Data

```

N <- 100
J <- 5 #Number of levels in factor (treatment) 1
K <- 3 #Number of levels in factor (treatment) 2
X <- matrix(cbind(round(runif(N, 0.5, J+0.49)),round(runif(N,0.5,K+0.49))),
            N, 2)
alpha <- runif(1,-1,1)
beta <- runif(J,-2,2)
beta[J] <- -sum(beta[1:(J-1)])
gamma <- runif(K,-2,2)
gamma[J] <- -sum(gamma[1:(K-1)])
y <- alpha + beta[X[,1]] + gamma[X[,2]] + rnorm(1,0,0.1)
mon.names <- c("LP","beta[5]","gamma[3]","sigma[1]","sigma[2]","sigma[3]",
              "s.beta","s.gamma","s.epsilon")
parm.names <- parm.names(list(alpha=0, beta=rep(0,J-1), gamma=rep(0,K-1),
                              log.sigma=rep(0,3)))
MyData <- list(J=J, K=K, N=N, X=X, mon.names=mon.names,
              parm.names=parm.names, y=y)

```

### 3.3. Initial Values

```
Initial.Values <- c(0, rep(0,(J-1)), rep(0,(K-1)), rep(log(1),3))
```

### 3.4. Model

```
Model <- function(parm, Data)
{
```

```

### Parameters
alpha <- parm[1]
beta <- rep(NA,Data$J)
beta[1:(Data$J-1)] <- parm[2:Data$J]
beta[J] <- -sum(beta[1:(Data$J-1)]) #Sum-to-zero constraint
gamma <- rep(NA,Data$K)
gamma[1:(Data$K-1)] <- parm[grepl("gamma", Data$parm.names)]
gamma[K] <- -sum(gamma[1:(Data$K-1)]) #Sum-to-zero constraint
sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
### Log(Prior Densities)
alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
beta.prior <- sum(dnorm(beta, 0, sigma[2], log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sigma[3], log=TRUE))
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
mu <- alpha + beta[Data$X[,1]] + gamma[Data$X[,2]]
LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
### Variance Components
s.beta <- sd(beta)
s.gamma <- sd(gamma)
s.epsilon <- sd(Data$y - mu)
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + gamma.prior +
      sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, beta[Data$J],
      gamma[Data$K], sigma, s.beta, s.gamma, s.epsilon), yhat=mu,
      parm=parm)
return(Modelout)
}

```

## 4. Approximate Bayesian Computation (ABC)

Approximate Bayesian Computation (ABC), also called likelihood-free estimation, is not a statistical method, but a family of numerical approximation techniques in Bayesian inference. ABC is especially useful when evaluation of the likelihood,  $p(\mathbf{y}|\Theta)$  is computationally prohibitive, or when suitable likelihoods are unavailable. The current example is the application of ABC in the context of linear regression. The log-likelihood is replaced with the negative sum of the distance between  $\mathbf{y}$  and  $\mathbf{y}^{rep}$  as the approximation of the log-likelihood. Distance reduces to the absolute difference. Although linear regression has an easily calculated likelihood, it is used as an example due to its generality. This example demonstrates how ABC may be estimated either with MCMC via the `LaplacesDemon` function or with Laplace Approximation via the `LaplaceApproximation` function. In this method, a tolerance (which is found often in ABC) does not need to be specified, and the logarithm of the unnormalized joint posterior density is maximized, as usual. The negative and summed distance, above, may be replaced with the negative and summed distance between summaries of the data,

rather than the data itself, but this has not been desirable in testing.

#### 4.1. Form

$$\mathbf{y} = \mu + \epsilon$$

$$\mu = \mathbf{X}\beta$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

#### 4.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,J)))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

#### 4.3. Initial Values

```
Initial.Values <- c(rep(0,J))
```

#### 4.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood Approximation
  mu <- tcrossprod(beta, Data$X)
  LL <- -sum(abs(Data$y - mu))
  ### Log-Posterior Approximation
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
  return(Modelout)
}
```

## 5. ARCH-M(1,1)

### 5.1. Form

$$\begin{aligned}
\mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma_t^2), \quad t = 1, \dots, T \\
\mathbf{y}^{new} &\sim \mathcal{N}(\mu_{T+1}, \sigma_{new}^2) \\
\mu_t &= \alpha + \phi \mathbf{y}_{t-1} + \delta \sigma_{t-1}^2, \quad t = 1, \dots, (T+1) \\
\epsilon_t &= \mathbf{y}_t - \mu_t \\
\alpha &\sim \mathcal{N}(0, 1000) \\
\phi &\sim \mathcal{N}(0, 1000) \\
\delta &\sim \mathcal{N}(0, 1000) \\
\sigma_{new}^2 &= \theta_1 + \theta_2 \epsilon_T^2 \\
\sigma_t^2 &= \theta_1 + \theta_2 \epsilon_{t-1}^2 \\
\theta_k &= \frac{1}{1 + \exp(-\theta_k)}, \quad k = 1, \dots, 2 \\
\theta_k &\sim \mathcal{N}(0, 1000) \in [-10, 10], \quad k = 1, \dots, 2
\end{aligned}$$

### 5.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,

```

```

0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
-0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
-0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)
T <- length(y)
mon.names <- c("LP", "ynew", "sigma2.new")
parm.names <- c("alpha", "phi", "delta", "logit.theta[1]", "logit.theta[2]")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 5.3. Initial Values

```
Initial.Values <- c(rep(0,3), rep(0.5,2))
```

### 5.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]; phi <- parm[2]; delta <- parm[3]
  theta <- invlogit(interval(parm[grep("logit.theta",
    Data$parm.names)], -10, 10))
  parm[grep("logit.theta", Data$parm.names)] <- logit(theta)
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)
  delta.prior <- dnorm(delta, 0, sqrt(1000), log=TRUE)
  theta.prior <- sum(dnorm(theta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- c(alpha, alpha + phi*Data$y[-Data$T])
  epsilon <- Data$y - mu
  sigma2 <- c(theta[1], theta[1] + theta[2]*epsilon[-Data$T]^2)
  mu <- mu + delta*sigma2
  ynew <- alpha + phi*Data$y[Data$T] + delta*sigma2[Data$T]
  sigma2.new <- theta[1] + theta[2]*epsilon[Data$T]^2
  LL <- sum(dnorm(Data$y, mu, sqrt(sigma2), log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + phi.prior + delta.prior + theta.prior +
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, ynew, sigma2.new),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 6. Autoregression, AR(1)

### 6.1. Form

$$\begin{aligned}
 \mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma^2), \quad t = 1, \dots, T \\
 \mathbf{y}^{new} &= \alpha + \mu_{T+1} \\
 \mu_t &= \alpha + \phi \mathbf{y}_{t-1}, \quad t = 1, \dots, (T+1) \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \phi &\sim \mathcal{N}(0, 1000) \\
 \sigma &\sim \mathcal{HC}(25)
 \end{aligned}$$

### 6.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
      0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
      0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
      -0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
      0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
      -0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
      0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)
T <- length(y)
mon.names <- c("LP", "sigma", "ynew")

```

```

parm.names <- c("alpha","phi","log.sigma")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 6.3. Initial Values

```
Initial.Values <- c(rep(0,2), log(1))
```

### 6.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]; phi <- parm[2]; sigma <- exp(parm[3])
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- c(alpha, alpha + phi*Data$y[-Data$T])
  ynew <- alpha + phi*Data$y[Data$T]
  LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + phi.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,ynew),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 7. Autoregressive Conditional Heteroskedasticity, ARCH(1,1)

### 7.1. Form

$$\begin{aligned}
\mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma_t^2), \quad t = 1, \dots, T \\
\mathbf{y}^{new} &\sim \mathcal{N}(\mu_{T+1}, \sigma_{new}^2) \\
\mu_t &= \alpha + \phi \mathbf{y}_{t-1}, \quad t = 1, \dots, (T+1) \\
\epsilon_t &= \mathbf{y}_t - \mu_t \\
\alpha &\sim \mathcal{N}(0, 1000) \\
\phi &\sim \mathcal{N}(0, 1000) \\
\sigma_{new}^2 &= \theta_1 + \theta_2 \epsilon_T^2 \\
\sigma_t^2 &= \theta_1 + \theta_2 \epsilon_{t-1}^2,
\end{aligned}$$

$$\theta_1 \sim \mathcal{N}(0, 1000) \in [0, \infty]$$

$$\theta_2 \sim \mathcal{U}(1.0E - 100, 1)$$

## 7.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
      0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
      0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
      -0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
      0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
      -0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
      0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)
T <- length(y)
mon.names <- c("LP", "ynew", "sigma2.new")
parm.names <- c("alpha", "phi", "logit.theta[1]", "logit.theta[2]")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

## 7.3. Initial Values

```
Initial.Values <- c(rep(0,2), rep(0.5,2))
```

## 7.4. Model

```

Model <- function(parm, Data)
{

```

```

### Parameters
alpha <- parm[1]; phi <- parm[2]
theta <- invlogit(interval(parm[grep("logit.theta",
  Data$parm.names)], -10, 10))
parm[grep("logit.theta", Data$parm.names)] <- logit(theta)
### Log(Prior Densities)
alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)
theta.prior <- sum(dnorm(theta, 0, sqrt(1000), log=TRUE))
### Log-Likelihood
mu <- c(alpha, alpha + phi*Data$y[-Data$T])
ynew <- alpha + phi*Data$y[Data$T]
epsilon <- Data$y - mu
sigma2 <- c(theta[1], theta[1] + theta[2]*epsilon[-Data$T]^2)
sigma2.new <- theta[1] + theta[2]*epsilon[Data$T]^2
LL <- sum(dnorm(Data$y, mu, sqrt(sigma2), log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + phi.prior + theta.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, ynew,
  sigma2.new), yhat=mu, parm=parm)
return(Modelout)
}

```

## 8. Autoregressive Moving Average, ARMA(1,1)

### 8.1. Form

$$\begin{aligned}
 \mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma^2), \quad t = 1, \dots, T \\
 \mathbf{y}^{new} &= \alpha + \phi \mathbf{y}_T + \theta \epsilon_T \\
 \mu_t &= \alpha + \phi \mathbf{y}_{t-1} + \theta \epsilon_{t-1} \\
 \epsilon_t &= \mathbf{y}_t - \mu_t \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \phi &\sim \mathcal{N}(0, 1000) \\
 \sigma &\sim \mathcal{HC}(25) \\
 \theta &\sim \mathcal{N}(0, 1000)
 \end{aligned}$$

### 8.2. Data

### 8.3. Data

```
y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
```

```

2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
-0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
-0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
-0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
-0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
-0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
-0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
-0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
-0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
-0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
-0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
-0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)

```

```

T <- length(y)
mon.names <- c("LP", "sigma", "ynew")
parm.names <- c("alpha", "phi", "sigma", "theta")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

## 8.4. Initial Values

```
Initial.Values <- c(rep(0,2), 0, log(1))
```

## 8.5. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]; phi <- parm[2]; theta <- parm[3]
  sigma <- exp(parm[4])
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)

```

```

sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
theta.prior <- dnorm(theta, 0, sqrt(1000), log=TRUE)
### Log-Likelihood
mu <- c(alpha, alpha + phi*Data$y[-Data$T])
epsilon <- Data$y - mu
mu <- c(mu[1], mu[-1] + theta * epsilon[-Data$T])
ynew <- alpha + phi*Data$y[Data$T] + theta*epsilon[Data$T]
LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + phi.prior + sigma.prior + theta.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, sigma, ynew),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 9. Beta Regression

### 9.1. Form

$$y \sim \mathcal{BETA}(a, b)$$

$$a = \mu\phi$$

$$b = (1 - \mu)\phi$$

$$\mu = \Phi(\beta_1 + \beta_2 \mathbf{x})$$

$$\beta_j \sim \mathcal{N}(0, 10), \quad j = 1, \dots, J$$

$$\phi \sim \mathcal{G}(1, 1)$$

where  $\Phi$  is the normal CDF.

### 9.2. Data

```

N <- 10
x <- runif(N)
y <- qbeta(0.5, pnorm(2-3*x)*4, (1-pnorm(2-3*x))*4)
mon.names <- "LP"
parm.names <- c("beta[1]", "beta[2]", "log.phi")
MyData <- list(x=x, y=y, mon.names=mon.names, parm.names=parm.names)

```

### 9.3. Initial Values

```
Initial.Values <- c(rep(0,2), log(0.01))
```

## 9.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:2]; phi <- exp(parm[3])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(10), log=TRUE))
  phi.prior <- dgamma(phi, 1, 1, log=TRUE)
  ### Log-Likelihood
  mu <- pnorm(beta[1] + beta[2]*Data$x)
  a <- mu * phi
  b <- (1-mu) * phi
  LL <- sum(dbeta(Data$y, a, b, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + phi.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
  return(Modelout)
}
```

## 10. Beta-Binomial

### 10.1. Form

$$y_i \sim \mathcal{BIN}(\mathbf{n}_i, \pi_i), \quad i = 1, \dots, N$$

$$\pi_i \sim \mathcal{BETA}(\alpha, \beta) \in [0.001, 0.999]$$

### 10.2. Data

```
N <- 20
n <- round(runif(N, 50, 100))
y <- round(runif(N, 1, 10))
mon.names <- "LP"
parm.names <- parm.names(list(pi=rep(0,N)))
MyData <- list(N=N, mon.names=mon.names, n=n, parm.names=parm.names, y=y)
```

### 10.3. Initial Values

```
Initial.Values <- c(rep(0.5,N))
```

### 10.4. Model

```
Model <- function(parm, Data)
{
```

```

### Parameters
pi <- interval(parm[1:Data$N], 0.001, 0.999)
parm[1:Data$N] <- pi
### Log(Prior Densities)
pi.prior <- sum(dbeta(pi, 1, 1, log=TRUE))
### Log-Likelihood
LL <- sum(dbinom(Data$y, Data$n, pi, log=TRUE))
yrep <- pi * Data$n
### Log-Posterior
LP <- LL + pi.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
return(Modelout)
}

```

## 11. Binary Logit

### 11.1. Form

$$\mathbf{y} \sim \mathcal{BERN}(\eta)$$

$$\eta = \frac{1}{1 + \exp(-\mu)}$$

$$\mu = \mathbf{X}\beta$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

### 11.2. Data

```

data(demonsnacks)
N <- NROW(demonsnacks)
J <- 3
y <- ifelse(demonsnacks$Calories <= 137, 0, 1)
X <- cbind(1, as.matrix(demonsnacks[,c(7,8)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,J)))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 11.3. Initial Values

```
Initial.Values <- rep(0,J)
```

### 11.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  mu <- tcrossprod(beta, Data$X)
  eta <- invlogit(mu)
  ### Log-Likelihood
  LL <- sum(dbern(Data$y, eta, log=TRUE))
  yrep <- ifelse(eta >= (sum(Data$y)/length(Data$y)),1,0)
  ### Log-Posterior
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP,
    yhat=yrep, parm=parm)
  return(Modelout)
}
```

## 12. Binary Log-Log Link Mixture

A weighted mixture of the log-log and complementary log-log link functions is used, where  $\alpha$  is the weight. Since the log-log and complementary log-log link functions are asymmetric (as opposed to the symmetric logit and probit link functions), it may be unknown *a priori* whether the log-log or complementary log-log will perform better.

### 12.1. Form

$$\begin{aligned} \mathbf{y} &\sim \mathcal{BERN}(\eta) \\ \eta &= \alpha \exp(-\exp(\mu)) + (1 - \alpha)(1 - \exp(-\exp(\mu))) \\ \mu &= \mathbf{X}\beta \\ \alpha &\sim \mathcal{U}(0, 1) \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \end{aligned}$$

### 12.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- 3
y <- ifelse(demonsnacks$Calories <= 30, 0, 1)
X <- cbind(1, as.matrix(demonsnacks[,c(7,8)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
```

```

mon.names <- c("LP","alpha")
parm.names <- parm.names(list(beta=rep(0,J), logit.alpha=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 12.3. Initial Values

```
Initial.Values <- c(rep(0,J), 0)
```

### 12.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- invlogit(parm[Data$J+1])
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  alpha.prior <- dunif(alpha, 0, 1, log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  mu <- tcrossprod(beta, Data$X)
  eta <- alpha*invloglog(mu) + (1-alpha)*invcloglog(mu)
  ### Log-Likelihood
  LL <- sum(dbern(Data$y, eta, log=TRUE))
  yrep <- ifelse(eta >= (sum(Data$y)/length(Data$y)),1,0)
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,alpha),
    yhat=yrep, parm=parm)
  return(Modelout)
}

```

## 13. Binary Probit

### 13.1. Form

$$\mathbf{y} \sim \mathcal{BERN}(\mathbf{p})$$

$$\mathbf{p} = \phi(\boldsymbol{\mu})$$

$$\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta} \in [-10, 10]$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

where  $\phi$  is the inverse CDF, and  $J=3$ .

### 13.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- 3
y <- ifelse(demonsnacks$Calories <= 137, 0, 1)
X <- cbind(1, as.matrix(demonsnacks[,c(7,8)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,J)))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

### 13.3. Initial Values

```
Initial.Values <- rep(0,J)
```

### 13.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  mu <- interval(mu, -10, 10)
  p <- pnorm(mu)
  LL <- sum(dbern(Data$y, p, log=TRUE))
  yrep <- ifelse(p >= (sum(Data$y)/length(Data$y)),1,0)
  ### Log-Posterior
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
  return(Modelout)
}
```

## 14. Binomial Logit

### 14.1. Form

$$\mathbf{y} \sim \mathcal{BIN}(\mathbf{p}, \mathbf{n})$$

$$\mathbf{p} = \frac{1}{1 + \exp(-\boldsymbol{\mu})}$$

$$\boldsymbol{\mu} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 \mathbf{x}$$

$$\boldsymbol{\beta}_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

## 14.2. Data

```
#10 Trials
exposed <- c(100,100,100,100,100,100,100,100,100,100)
deaths <- c(10,20,30,40,50,60,70,80,90,100)
dose <- c(1,2,3,4,5,6,7,8,9,10)
J <- 2 #Number of parameters
mon.names <- "LP"
parm.names <- c("beta[1]","beta[2]")
MyData <- list(J=J, n=exposed, mon.names=mon.names, parm.names=parm.names,
               x=dose, y=deaths)
```

## 14.3. Initial Values

```
Initial.Values <- rep(0,J)
```

## 14.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- beta[1] + beta[2]*Data$x
  p <- invlogit(mu)
  LL <- sum(dbinom(Data$y, Data$n, p, log=TRUE))
  yrep <- p * Data$n
  ### Log-Posterior
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
  return(Modelout)
}
```

# 15. Binomial Probit

## 15.1. Form

$$\mathbf{y} \sim \mathcal{BIN}(\mathbf{p}, \mathbf{n})$$

$$\mathbf{p} = \phi(\boldsymbol{\mu})$$

$$\boldsymbol{\mu} = \beta_1 + \beta_2 \mathbf{x} \in [-10, 10]$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

where  $\phi$  is the inverse CDF, and  $J=2$ .

## 15.2. Data

```
#10 Trials
exposed <- c(100,100,100,100,100,100,100,100,100,100)
deaths <- c(10,20,30,40,50,60,70,80,90,100)
dose <- c(1,2,3,4,5,6,7,8,9,10)
J <- 2 #Number of parameters
mon.names <- "LP"
parm.names <- c("beta[1]","beta[2]")
MyData <- list(J=J, n=exposed, mon.names=mon.names, parm.names=parm.names,
               x=dose, y=deaths)
```

## 15.3. Initial Values

```
Initial.Values <- rep(0,J)
```

## 15.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- beta[1] + beta[2]*Data$x
  mu <- interval(mu, -10, 10)
  p <- pnorm(mu)
  LL <- sum(dbinom(Data$y, Data$n, p, log=TRUE))
  yrep <- p * Data$n
  ### Log-Posterior
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep,
                  parm=parm)
  return(Modelout)
}
```

# 16. Cluster Analysis

This is a parametric model-based cluster analysis, also called a finite mixture model or latent class cluster analysis.

## 16.1. Form

$$\mathbf{Y}_{i,j} \sim \mathcal{N}(\mu_{\theta[i],j}, \sigma_{\theta[i]}^2), \quad i = 1, \dots, N, \quad j = 1, \dots, J$$

$$\theta_i = \text{Max}(\mathbf{p}_{i,1:C})$$

$$\mathbf{p}_{i,c} = \frac{\delta_{i,c}}{\sum_{c=1}^C \delta_{i,c}}$$

$$\pi_{1:C} \sim \mathcal{D}(\alpha_{1:C})$$

$$\pi_c = \frac{\sum_{i=1}^N \delta_{i,c}}{\sum \delta}$$

$$\alpha_c = 1$$

$$\delta_{i,C} = 1$$

$$\delta_{i,c} \sim \mathcal{N}(\log(\frac{1}{C}), 1000) \in [\exp(-10), \exp(10)], \quad c = 1, \dots, (C-1)$$

$$\mu_{c,j} \sim \mathcal{N}(0, \nu_j^2)$$

$$\sigma_c \sim \mathcal{HC}(25)$$

$$\nu_j \sim \mathcal{HC}(25)$$

## 16.2. Data

```

C <- 3 #Number of clusters
alpha <- rep(1,C) #Prior probability of cluster proportion
# Create a Y matrix
n <- 100; N <- 15 #Full sample; model sample
J <- 5 #Number of predictor variables
cluster <- round(runif(n,0.5,C+0.49))
centers <- matrix(runif(C*J, 0, 10), C, J)
Y.Full <- matrix(0, n, J)
for (i in 1:n) {for (j in 1:J)
  {Y.Full[i,j] <- rnorm(1,centers[cluster[i],j],1)}}
mean.temp <- colMeans(Y.Full)
sigma.temp <- apply(Y.Full,2,sd)
centers.cs <- (centers - matrix(rep(mean.temp,C), C, J, byrow=TRUE)) /
  (2 * matrix(rep(sigma.temp,C), C, J, byrow=TRUE))
for (j in 1:J) {Y.Full[,j] <- scale(Y.Full[,j],2)}
#summary(Y.Full)
MySample <- sample(1:n, N)
Y <- Y.Full[MySample,]
mon.names <- c("LP", parm.names(list(nu=rep(0,J), pi=rep(0,C),
  sigma=rep(0,C), theta=rep(0,N))))
parm.names <- parm.names(list(log.delta=matrix(0,N,C-1), mu=matrix(0,C,J),
  log.nu=rep(0,J), log.sigma=rep(0,C)))
MyData <- list(C=C, J=J, N=N, Y=Y, alpha=alpha, mon.names=mon.names,
  parm.names=parm.names)

```

### 16.3. Initial Values

```
Initial.Values <- c(runif(N*(C-1),-1,1), rep(0,C*J), rep(0,J), rep(0,C))
```

### 16.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  delta <- interval(parm[grepl("log.delta", Data$parm.names)], -10, 10)
  parm[grepl("log.delta", Data$parm.names)] <- delta
  delta <- matrix(c(exp(delta), rep(1, Data$N)), Data$N, Data$C)
  mu <- matrix(parm[grepl("mu", Data$parm.names)], Data$C, Data$J)
  nu <- exp(parm[grepl("log.nu", Data$parm.names)])
  pi <- colSums(delta) / sum(delta)
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  delta.prior <- sum(dtrunc(delta, "norm", a=exp(-10), b=exp(10),
    mean=log(1/Data$C), sd=sqrt(1000), log=TRUE))
  mu.prior <- sum(dnorm(mu, 0, matrix(rep(nu, Data$C), Data$C,
    Data$J, byrow=TRUE), log=TRUE))
  nu.prior <- sum(dhalfcauchy(nu, 25, log=TRUE))
  pi.prior <- ddirichlet(pi, Data$alpha, log=TRUE)
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  p <- delta / rowSums(delta)
  theta <- apply(p, 1, which.max)
  LL <- sum(dnorm(Data$Y, mu[theta,], sigma[theta,], log=TRUE))
  Yrep <- mu[theta,]
  ### Log-Posterior
  LP <- LL + delta.prior + mu.prior + nu.prior + pi.prior +
    sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, nu, pi, sigma, theta),
    yhat=Yrep, parm=parm)
  return(Modelout)
}
```

## 17. Conditional Autoregression (CAR), Poisson

This CAR example is a slightly modified form of example 7.3 (Model A) in [Congdon \(2003\)](#). The Scottish lip cancer data also appears in the WinBUGS ([Spiegelhalter, Thomas, Best, and Lunn 2003](#)) examples and is a widely analyzed example. The data  $\mathbf{y}$  consists of counts for  $i = 1, \dots, 56$  counties in Scotland. A single predictor  $\mathbf{x}$  is provided. The errors,  $\epsilon$ , are allowed to include spatial effects as smoothing by spatial effects from areal neighbors. Interactions  $\mathbf{w}$  between counties are in terms of dummy indicators for contiguity (areal neighbors). The list of  $NN$  areal neighbors is in the *adj* variable, and cumulative positions are in variable *C*. The

vector  $\epsilon_\mu$  is the mean of each area's error, and is a weighted average of errors in contiguous areas.

### 17.1. Form

$$\begin{aligned}\mathbf{y} &\sim \mathcal{P}(\lambda) \\ \lambda &= \exp(\log(\mathbf{E}) + \beta_1 + \beta_2 \mathbf{x} + \epsilon) \\ \epsilon &\sim \mathcal{N}(\epsilon_\mu, \sigma^2) \\ \epsilon_{\mu[i]} &= \rho \sum_{j=1}^J \mathbf{w}_{i,j} \epsilon_j, \quad i = 1, \dots, N \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \rho &\sim \mathcal{U}(-1, 1) \\ \sigma &\sim \mathcal{HC}(25)\end{aligned}$$

### 17.2. Data

```
N <- 56 #Number of areas
NN <- 264 #Number of adjacent areas
y <- c(9,39,11,9,15,8,26,7,6,20,13,5,3,8,17,9,2,7,9,7,16,31,11,7,19,15,7,
      10,16,11,5,3,7,8,11,9,11,8,6,4,10,8,2,6,19,3,2,3,28,6,1,1,1,1,0,0)
E <- c( 1.4,8.7,3.0,2.5,4.3,2.4,8.1,2.3,2.0,6.6,4.4,1.8,1.1,3.3,7.8,4.6,
      1.1,4.2,5.5,4.4,10.5,22.7,8.8,5.6,15.5,12.5,6.0,9.0,14.4,10.2,4.8,
      2.9,7.0,8.5,12.3,10.1,12.7,9.4,7.2,5.3,18.8,15.8,4.3,14.6,50.7,8.2,
      5.6,9.3,88.7,19.6,3.4,3.6,5.7,7.0,4.2,1.8) #Expected
x <- c(16,16,10,24,10,24,10,7,7,16,7,16,10,24,7,16,10,7,7,10,7,16,10,7,1,1,
      7,7,10,10,7,24,10,7,7,0,10,1,16,0,1,16,16,0,1,7,1,1,0,1,1,0,1,1,16,10)
adj <- c(5,9,11,19, #Area 1 is adjacent to areas 5, 9, 11, and 19
      7,10, #Area 2 is adjacent to areas 7 and 10
      6,12,
      18,20,28,
      1,11,12,13,19,
      3,8,
      2,10,13,16,17,
      6,
      1,11,17,19,23,29,
      2,7,16,22,
      1,5,9,12,
      3,5,11,
      5,7,17,19,
      31,32,35,
      25,29,50,
      7,10,17,21,22,29,
```

```

7,9,13,16,19,29,
4,20,28,33,55,56,
1,5,9,13,17,
4,18,55,
16,29,50,
10,16,
9,29,34,36,37,39,
27,30,31,44,47,48,55,56,
15,26,29,
25,29,42,43,
24,31,32,55,
4,18,33,45,
9,15,16,17,21,23,25,26,34,43,50,
24,38,42,44,45,56,
14,24,27,32,35,46,47,
14,27,31,35,
18,28,45,56,
23,29,39,40,42,43,51,52,54,
14,31,32,37,46,
23,37,39,41,
23,35,36,41,46,
30,42,44,49,51,54,
23,34,36,40,41,
34,39,41,49,52,
36,37,39,40,46,49,53,
26,30,34,38,43,51,
26,29,34,42,
24,30,38,48,49,
28,30,33,56,
31,35,37,41,47,53,
24,31,46,48,49,53,
24,44,47,49,
38,40,41,44,47,48,52,53,54,
15,21,29,
34,38,42,54,
34,40,49,54,
41,46,47,49,
34,38,49,51,52,
18,20,24,27,56,
18,24,30,33,45,55)
# C has length N+1 and refers to cumulative position (-1) in the adj
# variable. For example, area 1 begins at 0 (position 1-1), and
# area 2 begins at 4 (position 5-1), etc.
C <- c(0,4,6,8,11,16,18,23,24,30,34,38,41,45,48,51,57,63,69,74,77,80,82,
      88,96,99,103,107,111,122,128,135,139,143,152,157,161,166,172,177,182,
      189,195,199,204,208,214,220,224,233,236,240,244,248,253,258,264)
mon.names <- c("LP","sigma")

```

```

parm.names <- parm.names(list(beta=rep(0,2), epsilon=rep(0,N), rho=0,
  log.sigma=0))
MyData <- list(C=C, E=E, N=N, NN=NN, adj=adj, mon.names=mon.names,
  parm.names=parm.names, x=x, y=y)

```

### 17.3. Initial Values

```
Initial.Values <- c(rep(0,2), rep(0,N), 0, 0)
```

### 17.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:2]
  epsilon <- parm[grep("epsilon", Data$parm.names)]
  rho <- interval(parm[grep("rho", Data$parm.names)], -1, 1)
  parm[grep("rho", Data$parm.names)] <- rho
  w <- epsilon[Data$adj]
  epsilon.mu <- epsilon
  for (i in 1:N) {
    epsilon.mu[i] <- rho * sum(w[(Data$C[i]+1):(Data$C[i+1])]))
  }
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  epsilon.prior <- sum(dnorm(epsilon, epsilon.mu, sigma,
    log=TRUE))
  rho.prior <- dunif(rho, -1, 1, log=TRUE)
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  lambda <- exp(log(Data$E) + beta[1] + beta[2]*Data$x/10 + epsilon)
  LL <- sum(dpois(Data$y, lambda, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + epsilon.prior + rho.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma), yhat=lambda,
    parm=parm)
  return(Modelout)
}

```

## 18. Conditional Predictive Ordinate

For a more complete introduction to the conditional predictive ordinate (CPO), see the vignette entitled “Bayesian Inference”. Following is a brief guide to the applied use of CPO.

To include CPO in any model that is to be updated with MCMC, calculate and monitor the record-level inverse of the likelihood,  $\text{InvL}_i$  for records  $i = 1, \dots, N$ .  $\text{CPO}_i$  is the inverse of the posterior mean of  $\text{InvL}_i$ . The inverse  $\text{CPO}_i$ , or  $\text{ICPO}_i$ , is the posterior mean of  $\text{InvL}_i$ .

ICPOs larger than 40 can be considered as possible outliers, and higher than 70 as extreme values.

Here, CPO is added to the linear regression example in section 37. In this data, record 6 is a possible outlier, and record 8 is an extreme value.

### 18.1. Form

$$\begin{aligned} \mathbf{y} &\sim \mathcal{N}(\mu, \sigma^2) \\ \mu &= \mathbf{X}\beta \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \sigma &\sim \mathcal{HC}(25) \end{aligned}$$

### 18.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- c("LP", "sigma", parm.names(list(InvL=rep(0,N))))
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

### 18.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 18.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- dnorm(Data$y, mu, sigma, log=TRUE)
  InvL <- 1 / exp(LL)
  LL <- sum(LL)
  ### Log-Posterior
```

```

LP <- LL + beta.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,InvL),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 19. Contingency Table

The two-way contingency table, matrix  $\mathbf{Y}$ , can easily be extended to more dimensions. For this example, it is vectorized as  $y$ , and used like an ANOVA data set. Contingency table  $\mathbf{Y}$  has  $J$  rows and  $K$  columns. The cell counts are fit with Poisson regression, according to intercept  $\alpha$ , main effects  $\beta_j$  for each row, main effects  $\gamma_k$  for each column, and interaction effects  $\delta_{j,k}$  for dependence effects. An omnibus (all cells) test of independence is done by estimating two models (one with  $\delta$ , and one without), and a large enough Bayes Factor indicates a violation of independence when the model with  $\delta$  fits better than the model without  $\delta$ . In an ANOVA-like style, main effects contrasts can be used to distinguish rows or groups of rows from each other, as well as with columns. Likewise, interaction effects contrasts can be used to test independence in groups of  $\delta_{j,k}$  elements. Finally, single-cell interactions can be used to indicate violations of independence for a given cell, such as when zero is not within its 95% probability interval. Although a little different, this example is similar to a method presented by [Albert \(1997\)](#).

### 19.1. Form

$$\begin{aligned}
 \mathbf{Y}_{j,k} &\sim \mathcal{P}(\lambda_{j,k}), \quad j = 1, \dots, J, \quad k = 1, \dots, K \\
 \lambda_{j,k} &= \exp(\alpha + \beta_j + \gamma_k + \delta_{j,k}), \quad j = 1, \dots, J, \quad k = 1, \dots, K \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \beta_j &\sim \mathcal{N}(0, \beta_\sigma^2), \quad j = 1, \dots, J \\
 \beta_\sigma &\sim \mathcal{HC}(25) \\
 \gamma_k &\sim \mathcal{N}(0, \gamma_\sigma^2), \quad k = 1, \dots, K \\
 \gamma_\sigma &\sim \mathcal{HC}(25) \\
 \delta_{j,k} &\sim \mathcal{N}(0, \delta_\sigma^2) \\
 \delta_\sigma &\sim \mathcal{HC}(25)
 \end{aligned}$$

### 19.2. Data

```

J <- 4 #Rows
K <- 4 #Columns
Y <- matrix(c(10,20,60,20, 40,30,10,40, 10,10,40,10, 40,50,1,40), J, K,
  dimnames=list(c("Chrysler","Ford","Foreign","GM"),

```

```

      c("I-4","I-6","V-6","V-8"))))
y <- as.vector(Y)
N <- length(y) #Cells
r <- rep(1:J, N/J)
c <- rep(1,K)
for (k in 2:K) {c <- c(c, rep(k, K))}
mon.names <- c("LP","beta.sigma","gamma.sigma","delta.sigma")
parm.names <- parm.names(list(alpha=0, beta=rep(0,J), gamma=rep(0,J),
  log.b.sigma=0, log.g.sigma=0, log.d.sigma=0,
  delta=matrix(0,J,K)))
MyData <- list(J=J, K=K, N=N, c=c, mon.names=mon.names,
  parm.names=parm.names, r=r, y=y)

```

### 19.3. Initial Values

```
Initial.Values <- c(0, rep(0,J), rep(0,K), rep(0,3), rep(0,J*K))
```

### 19.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  beta.sigma <- exp(parm[grepl("log.b.sigma", Data$parm.names)])
  gamma.sigma <- exp(parm[grepl("log.g.sigma", Data$parm.names)])
  delta.sigma <- exp(parm[grepl("log.d.sigma", Data$parm.names)])
  ### Parameters
  alpha <- parm[grepl("alpha", Data$parm.names)]
  beta <- parm[grepl("beta", Data$parm.names)]
  gamma <- parm[grepl("gamma", Data$parm.names)]
  delta <- matrix(parm[grepl("delta", Data$parm.names)],
    Data$J, Data$K)
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, beta.sigma, log=TRUE))
  beta.sigma.prior <- dhalfcauchy(beta.sigma, 25, log=TRUE)
  gamma.prior <- sum(dnorm(gamma, 0, gamma.sigma, log=TRUE))
  gamma.sigma.prior <- dhalfcauchy(gamma.sigma, 25, log=TRUE)
  delta.prior <- sum(dnorm(delta, 0, delta.sigma, log=TRUE))
  delta.sigma.prior <- dhalfcauchy(delta.sigma, 25, log=TRUE)
  ### Log-Likelihood
  lambda <- exp(alpha + beta[Data$r] + gamma[Data$c] +
    diag(delta[Data$r,Data$c]))
  LL <- sum(dpois(Data$y, lambda, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior + beta.sigma.prior +
    gamma.prior + gamma.sigma.prior + delta.prior +
    delta.sigma.prior

```

```

Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, beta.sigma,
      gamma.sigma, delta.sigma), yhat=lambda, parm=parm)
return(Modelout)
}

```

## 20. Discrete Choice, Conditional Logit

### 20.1. Form

$$\begin{aligned}
 \mathbf{y}_i &\sim \mathcal{CAT}(\mathbf{p}_{i,1:J}), \quad i = 1, \dots, N, \quad j = 1, \dots, J \\
 \mathbf{p}_{i,j} &= \frac{\phi_{i,j}}{\sum_{j=1}^J \phi_{i,j}} \\
 \phi &= \exp(\mu) \\
 \mu_{i,j} &= \beta_{j,1:K} \mathbf{X}_{i,1:K} + \gamma \mathbf{Z}_{i,1:C} \in [-700, 700], \quad j = 1, \dots, (J-1) \\
 \mu_{i,J} &= \gamma \mathbf{Z}_{i,1:C} \\
 \beta_{j,k} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1) \\
 \gamma_c &\sim \mathcal{N}(0, 1000)
 \end{aligned}$$

### 20.2. Data

```

y <- x01 <- x02 <- z01 <- z02 <- c(1:300)
y[1:100] <- 1
y[101:200] <- 2
y[201:300] <- 3
x01[1:100] <- rnorm(100, 25, 2.5)
x01[101:200] <- rnorm(100, 40, 4.0)
x01[201:300] <- rnorm(100, 35, 3.5)
x02[1:100] <- rnorm(100, 2.51, 0.25)
x02[101:200] <- rnorm(100, 2.01, 0.20)
x02[201:300] <- rnorm(100, 2.70, 0.27)
z01[1:100] <- 1
z01[101:200] <- 2
z01[201:300] <- 3
z02[1:100] <- 40
z02[101:200] <- 50
z02[201:300] <- 100
N <- length(y)
J <- 3 #Number of categories in y
K <- 3 #Number of individual attributes (including the intercept)
C <- 2 #Number of choice-based attributes (intercept is not included)
X <- matrix(c(rep(1,N),x01,x02),N,K) #Design matrix of individual attrib.

```

```

Z <- matrix(c(z01,z02),N,C) #Design matrix of choice-based attributes
mon.names <- "LP"
parm.names <- parm.names(list(beta=matrix(0,J-1,K), gamma=rep(0,C)))
MyData <- list(C=C, J=J, K=K, N=N, X=X, Z=Z, mon.names=mon.names,
               parm.names=parm.names, y=y)

```

### 20.3. Initial Values

```
Initial.Values <- c(rep(0,(J-1)*K), rep(0,C))
```

### 20.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$J-1, Data$K)
  gamma <- parm[grepl("gamma", Data$parm.names)]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  gamma.prior <- sum(dnorm(gamma, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- matrix(rep(tcrossprod(gamma, Data$Z),J),Data$N,Data$J)
  mu[,1] <- mu[,1] + tcrossprod(beta[1,], Data$X)
  mu[,2] <- mu[,2] + tcrossprod(beta[2,], Data$X)
  mu <- interval(mu, -700, 700)
  phi <- exp(mu)
  p <- phi / rowSums(phi)
  LL <- sum(dcat(Data$y, p, log=TRUE))
  yrep <- apply(p,1,which.max)
  ### Log-Posterior
  LP <- LL + beta.prior + gamma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
  return(Modelout)
}

```

## 21. Discrete Choice, Mixed Logit

### 21.1. Form

$$\mathbf{y}_i \sim \mathcal{CAT}(\mathbf{p}_{i,1:J}), \quad i = 1, \dots, N, \quad j = 1, \dots, J$$

$$\mathbf{p}_{i,j} = \frac{\phi_{i,j}}{\sum_{j=1}^J \phi_{i,j}}$$

$$\phi = \exp(\mu)$$

$$\mu_{i,j} = \beta_{j,1:K} \mathbf{X}_{i,1:K} + \gamma \mathbf{Z}_{i,1:C} \in [-700, 700], \quad j = 1, \dots, (J-1)$$

$$\mu_{i,J} = \gamma \mathbf{Z}_{i,1:C}$$

$$\beta_{j,k} \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1)$$

$$\gamma_c \sim \mathcal{N}(\zeta_{\mu[c]}, \zeta_{\sigma[c]}^2)$$

$$\zeta_{\mu[c]} \sim \mathcal{N}(0, 1000)$$

$$\zeta_{\sigma[c]} \sim \mathcal{HC}(25)$$

## 21.2. Data

```

y <- x01 <- x02 <- z01 <- z02 <- c(1:300)
y[1:100] <- 1
y[101:200] <- 2
y[201:300] <- 3
x01[1:100] <- rnorm(100, 25, 2.5)
x01[101:200] <- rnorm(100, 40, 4.0)
x01[201:300] <- rnorm(100, 35, 3.5)
x02[1:100] <- rnorm(100, 2.51, 0.25)
x02[101:200] <- rnorm(100, 2.01, 0.20)
x02[201:300] <- rnorm(100, 2.70, 0.27)
z01[1:100] <- 1
z01[101:200] <- 2
z01[201:300] <- 3
z02[1:100] <- 40
z02[101:200] <- 50
z02[201:300] <- 100
N <- length(y)
J <- 3 #Number of categories in y
K <- 3 #Number of individual attributes (including the intercept)
C <- 2 #Number of choice-based attributes (intercept is not included)
X <- matrix(c(rep(1,N),x01,x02),N,K) #Design matrix of individual attrib.
Z <- matrix(c(z01,z02),N,C) #Design matrix of choice-based attributes
mon.names <- c("LP", parm.names(list(zeta.sigma=rep(0,C))))
parm.names <- parm.names(list(beta=matrix(0,J-1,K), gamma=rep(0,C),
  zeta.mu=rep(0,C), log.zeta.sigma=rep(0,C)))
MyData <- list(C=C, J=J, K=K, N=N, X=X, Z=Z, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

## 21.3. Initial Values

```
Initial.Values <- c(rep(0,(J-1)*K), rep(0,N*C), rep(0,C), rep(0,C))
```

## 21.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters

```

```

beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$J-1, Data$K)
gamma <- parm[grepl("gamma", Data$parm.names)]
zeta.mu <- parm[grepl("zeta.mu", Data$parm.names)]
zeta.sigma <- exp(parm[grepl("log.zeta.sigma", Data$parm.names)])
### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sqrt(1000), log=TRUE))
zeta.mu.prior <- sum(dnorm(zeta.mu, 0, sqrt(1000), log=TRUE))
zeta.sigma.prior <- sum(dhalfcauchy(zeta.sigma, 25, log=TRUE))
### Log-Likelihood
mu <- matrix(rep(rowSums(gamma * Data$Z), J), Data$N, Data$J)
mu[,1] <- mu[,1] + tcrossprod(beta[1,], Data$X)
mu[,2] <- mu[,2] + tcrossprod(beta[2,], Data$X)
mu <- interval(mu, -700, 700)
phi <- exp(mu)
p <- phi / rowSums(phi)
LL <- sum(dcat(Data$y, p, log=TRUE))
yrep <- apply(p, 1, which.max)
### Log-Posterior
LP <- LL + beta.prior + gamma.prior + zeta.mu.prior + zeta.sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, zeta.sigma.prior),
  yhat=yrep, parm=parm)
return(Modelout)
}

```

## 22. Discrete Choice, Multinomial Probit

### 22.1. Form

$$\begin{aligned}
\mathbf{Z}_{i,1:J} &\sim \mathcal{N}_J(\mu_{i,1:J}, \Sigma), \quad i = 1, \dots, N \\
\mathbf{Z}_{i,j} &\in \begin{cases} [0,10] & \text{if } \mathbf{y}_i = j \\ [-10,0] & \end{cases} \\
\mu_{1:N,j} &= \mathbf{X}\beta_{j,1:K} + \mathbf{W}\gamma[a, 1 : C] \\
\mathbf{a} &= \begin{cases} 1 & \text{if } \mathbf{y}_i < J \\ 2 & \end{cases} \\
\Sigma &\sim \mathcal{IW}(J, \mathbf{R}), \quad \mathbf{R} = \mathbf{I}_J, \quad \Sigma[1,1] = 1 \\
\beta_{j,k} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1), \quad k = 1, \dots, K \\
\beta_{J,k} &= -\sum_{j=1}^{J-1} \beta_{j,k} \\
\gamma_{1,1:C} &\sim \mathcal{N}(0, 1000)
\end{aligned}$$

$$\gamma_{2,c} = -\gamma_{1,c}, \quad c = 1, \dots, C$$

$$\mathbf{Z}_{i,j} \sim \mathcal{N}(0, 1000) \in [-10, 10]$$

## 22.2. Data

```

y <- x1 <- x2 <- w1 <- w2 <- c(1:30)
y[1:10] <- 1
y[11:20] <- 2
y[21:30] <- 3
x1[1:10] <- rnorm(10, 25, 2.5)
x1[11:20] <- rnorm(10, 40, 4.0)
x1[21:30] <- rnorm(10, 35, 3.5)
x2[1:10] <- rnorm(10, 2.51, 0.25)
x2[11:20] <- rnorm(10, 2.01, 0.20)
x2[21:30] <- rnorm(10, 2.70, 0.27)
w1[1:10] <- 10
w1[11:20] <- 4
w1[21:30] <- 1
w2[1:10] <- 40
w2[11:20] <- 50
w2[21:30] <- 100
N <- length(y)
J <- length(unique(y)) #Number of categories in y
K <- 3 #Number of columns to be in design matrix X
R <- diag(J)
X <- matrix(c(rep(1,N),x1,x2),N,K)
C <- 2 #Number of choice-based attributes
W <- matrix(c(w1,w2),N,C) #Design matrix of choice-based attributes
mon.names <- "LP"
sigma.temp <- parm.names(list(Sigma=diag(J)), uppertri=1)
parm.names <- c(sigma.temp[2:length(sigma.temp)],
  parm.names(list(beta=matrix(0,(J-1),K), gamma=rep(0,C),
    Z=matrix(0,N,J))))
MyData <- list(J=J, K=K, N=N, R=R, W=W, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

## 22.3. Initial Values

```

Initial.Values <- c(rep(0,length(R[upper.tri(R, diag=TRUE)]-1),
  rep(0,(J-1)*K), rep(0,C), rep(0,N,J))

```

## 22.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters

```

```

beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$J-1, Data$K)
beta <- rbind(beta, colSums(beta)*-1) #Sum to zero constraint
gamma <- parm[grepl("gamma", Data$parm.names)]
gamma <- rbind(gamma, gamma*-1) #Sum to zero constraint
Sigma <- matrix(NA, Data$J, Data$J)
Sigma[upper.tri(Sigma, diag=TRUE)] <- c(0, parm[grepl("Sigma",
  Data$parm.names)])
Sigma[lower.tri(Sigma)] <- Sigma[upper.tri(Sigma)]
diag(Sigma) <- exp(diag(Sigma))
Z <- matrix(parm[grepl("Z", Data$parm.names)], Data$N, Data$J)
### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sqrt(1000), log=TRUE))
Sigma.prior <- dinvwishart(Sigma, Data$J, Data$R, log=TRUE)
Z.prior <- sum(dnorm(Z, 0, sqrt(1000), log=TRUE))
### Log-Likelihood
mu <- matrix(0, Data$N, Data$J)
mu <- matrix(c(rep(tcrossprod(gamma[1,], Data$W), J),
  tcrossprod(gamma[2,], Data$W)), Data$N, Data$J)
for (j in 1:Data$J) {mu[,j] <- mu[,j] + tcrossprod(beta[j,], Data$X)}
Y <- indmat(Data$y)
Z <- ifelse(Z > 10, 10, Z); Z <- ifelse({Y == 0} & {Z > 0}, 0, Z)
Z <- ifelse(Z < -10, -10, Z); Z <- ifelse({Y == 1} & {Z < 0}, 0, Z)
parm[grepl("Z", Data$parm.names)] <- as.vector(Z)
LL <- sum(dmvn(Z, mu, Sigma, log=TRUE))
yrep <- apply(Z, 1, which.max)
#eta <- exp(mu)
#p <- eta / rowSums(eta)
### Log-Posterior
LP <- LL + beta.prior + gamma.prior + Sigma.prior + Z.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
return(Modelout)
}

```

## 23. Distributed Lag, Koyck

This example applies Koyck or geometric distributed lags to  $k = 1, \dots, K$  discrete events in covariate  $\mathbf{x}$ , transforming the covariate into a  $N \times K$  matrix  $\mathbf{X}$  and creates a  $N \times K$  lag matrix  $\mathbf{L}$ .

### 23.1. Form

$$\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$$

$$\mu_t = \alpha + \phi \mathbf{y}_{t-1} + \sum_{k=1}^K \mathbf{X}_{t,k} \beta \lambda^{\mathbf{L}[t,k]}, \quad k = 1, \dots, K, \quad t = 2, \dots, T$$

$$\mu_1 = \alpha + \sum_{k=1}^K \mathbf{X}_{1,k} \beta \lambda^{\mathbf{L}[1,k]}, \quad k = 1, \dots, K$$

$$\alpha \sim \mathcal{N}(0, 1000)$$

$$\beta \sim \mathcal{N}(0, 1000)$$

$$\lambda \sim \mathcal{U}(0, 1)$$

$$\phi \sim \mathcal{N}(0, 1000)$$

$$\sigma \sim \mathcal{HC}(25)$$

### 23.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
      0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
      0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
      -0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
      0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
      -0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
      0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)

x <- c(0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
      1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,

```



}

## 24. Dynamic Linear Model (DLM)

The data is presented so that the time-series is subdivided into three sections: modeled ( $t = 1, \dots, T_m$ ), one-step ahead forecast ( $t = T_m + 1$ ), and future forecast [ $t = (T_m + 2), \dots, T$ ].

### 24.1. Form

$$\begin{aligned}
 \mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma_V^2), \quad t = 1, \dots, T_m \\
 \mathbf{y}_t^{new} &\sim \mathcal{N}(\mu_t, \sigma_V^2), \quad t = (T_m + 1), \dots, T \\
 \mu_t &= \alpha + \mathbf{x}_t \beta_t, \quad t = 1, \dots, T \\
 \alpha &\sim \mathcal{N}(0, 1000) \\
 \beta_1 &\sim \mathcal{N}(0, 1000) \\
 \beta_t &\sim \mathcal{N}(\beta_{t-1}, \sigma_W^2), \quad t = 2, \dots, T \\
 \sigma_V &\sim \mathcal{HC}(25) \\
 \sigma_W &\sim \mathcal{HC}(25)
 \end{aligned}$$

### 24.2. Data

```

T <- 20
T.m <- 14
beta.orig <- x <- rep(0,T)
for (t in 2:T) {
  beta.orig[t] <- beta.orig[t-1] + rnorm(1,0,0.1)
  x[t] <- x[t-1] + rnorm(1,0,0.1)}
y <- 10 + beta.orig*x + rnorm(T,0,0.1)
y[(T.m+2):T] <- NA
mon.names <- rep(NA, (T-T.m))
for (i in 1:(T-T.m)) mon.names[i] <- paste("mu[",(T.m+i),"]", sep="")
parm.names <- parm.names(list(alpha=0, beta=rep(0,T), log.beta.w.sigma=0,
  log.v.sigma=0))
MyData <- list(T=T, T.m=T.m, mon.names=mon.names, parm.names=parm.names,
  x=x, y=y)

```

### 24.3. Initial Values

```
Initial.Values <- rep(0,T+3)
```

## 24.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]
  beta <- parm[2:(Data$T+1)]
  beta.w.sigma <- exp(parm[Data$T+2])
  v.sigma <- exp(parm[Data$T+3])
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  beta.prior <- rep(0, Data$T)
  beta.prior[1] <- dnorm(beta[1], 0, sqrt(1000), log=TRUE)
  beta.prior[2:Data$T] <- dnorm(beta[2:Data$T], beta[1:(Data$T-1)],
    beta.w.sigma, log=TRUE)
  beta.w.sigma.prior <- dhalfcauchy(beta.w.sigma, 25, log=TRUE)
  v.sigma.prior <- dhalfcauchy(v.sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- alpha + beta*Data$x
  LL <- sum(dnorm(Data$y[1:Data$T.m], mu[1:Data$T.m], v.sigma,
    log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + sum(beta.prior) + beta.w.sigma.prior +
    v.sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=mu[(Data$T.m+1):Data$T],
    yhat=mu, parm=parm)
  return(Modelout)
}
```

## 25. Exponential Smoothing

### 25.1. Form

$$\begin{aligned}
 \mathbf{y} &\sim \mathcal{N}(\boldsymbol{\mu}, \sigma^2) \\
 \mu_t &= \alpha \mathbf{y}_{t-1} + (1 - \alpha) \mu_{t-1}, \quad t = 2, \dots, T \\
 \alpha &\sim \mathcal{U}(0, 1) \\
 \sigma &\sim \mathcal{HC}
 \end{aligned}$$

### 25.2. Data

```
T <- 10
y <- rep(0, T)
y[1] <- 0
for (t in 2:T) {y[t] <- y[t-1] + rnorm(1, 0, 0.1)}
```

```
mon.names <- c("LP", "sigma")
parm.names <- c("alpha", "log.sigma")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)
```

### 25.3. Initial Values

```
Initial.Values <- c(0.5, log(1))
```

### 25.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- interval(parm[1], 0, 1); parm[1] <- alpha
  sigma <- exp(parm[2])
  ### Log(Prior Densities)
  alpha.prior <- dunif(alpha, 0, 1, log=TRUE)
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- y
  mu[-1] <- alpha*Data$y[-1]
  mu[-1] <- mu[-1] + (1 - alpha) * mu[-Data$T]
  LL <- sum(dnorm(Data$y[-1], mu[-Data$T], sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma),
    yhat=mu, parm=parm)
  return(Modelout)
}
```

## 26. Factor Analysis, Confirmatory

Factor scores are in matrix  $\mathbf{F}$ , factor loadings for each variable are in vector  $\lambda$ , and  $\mathbf{f}$  is a vector that indicates which variable loads on which factor.

### 26.1. Form

$$\begin{aligned} \mathbf{Y}_{i,m} &\sim \mathcal{N}(\mu_{i,m}, \sigma_m^2), \quad i = 1, \dots, N, \quad m = 1, \dots, M \\ \mu_{i,m} &= \alpha_m + \lambda_m \mathbf{F}_{i,\mathbf{f}[m]}, \quad i = 1, \dots, N, \quad m = 1, \dots, M \\ \mathbf{F}_{i,1:P} &\sim \mathcal{N}_P(\gamma, \Omega^{-1}), \quad i = 1, \dots, N \\ \alpha_m &\sim \mathcal{N}(0, 1000), \quad m = 1, \dots, M \\ \lambda_m &\sim \mathcal{N}(0, 1000), \quad m = 1, \dots, M \end{aligned}$$

$$\sigma_m \sim \mathcal{HC}(25), \quad m = 1, \dots, M$$

$$\Omega \sim \mathcal{W}(N, S), \quad S = I_P$$

## 26.2. Data

```
data(swiss)
Y <- cbind(swiss$Agriculture, swiss$Examination, swiss$Education,
           swiss$Catholic, swiss$Infant.Mortality)
M <- NCOL(Y) #Number of variables
N <- NROW(Y) #Number of records
P <- 3 #Number of factors
f <- c(1,3,2,2,1) #Indicator f for the factor for each variable m
gamma <- rep(0,P)
S <- diag(P)
mon.names <- c("LP", "mu[1,1]")
parm.names <- parm.names(list(F=matrix(0,N,P), lambda=rep(0,M),
                               Omega=diag(P), alpha=rep(0,M), log.sigma=rep(0,M)),
                          uppertri=c(0,0,1,0,0))
MyData <- list(M=M, N=N, P=P, S=S, Y=Y, f=f, gamma=gamma,
              mon.names=mon.names, parm.names=parm.names)
```

## 26.3. Initial Values

```
Initial.Values <- c(rep(0,N*P), rep(0,M), S[upper.tri(S, diag=TRUE)],
                  rep(0,M), rep(0,M))
```

## 26.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[grepl("alpha", Data$parm.names)]
  lambda <- parm[grepl("lambda", Data$parm.names)]
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  F <- matrix(parm[grepl("F", Data$parm.names)], Data$N, Data$P)
  Omega <- matrix(NA, Data$P, Data$P)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[grepl("Omega",
    Data$parm.names)]
  Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
  Sigma <- solve(Omega)
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  lambda.prior <- sum(dnorm(lambda, 0, sqrt(1000), log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  Omega.prior <- dwishart(Omega, Data$N, Data$S, log=TRUE)
  F.prior <- sum(dmvn(F, Data$gamma, Sigma, log=TRUE))
  ### Log-Likelihood
```

```

mu <- Data$Y
for (m in 1:Data$M) {mu[,m] <- alpha[m] + lambda[m] * F[,Data$f[m]]}
LL <- sum(dnorm(Data$Y, mu, sigma, log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + lambda.prior + sigma.prior + F.prior +
      Omega.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,mu[1,1]),
                 yhat=mu, parm=parm)
return(Modelout)
}

```

## 27. Factor Analysis, Exploratory

Factor scores are in matrix  $\mathbf{F}$  and factor loadings are in matrix  $\Lambda$ . Although the calculation for the recommended number of factors to explore  $P$  is also provided below (Fokoue 2004), this example sets  $P = 3$ .

### 27.1. Form

$$\begin{aligned}
\mathbf{Y}_{i,m} &\sim \mathcal{N}(\mu_{i,m}, \sigma_m^2), \quad i = 1, \dots, N, \quad m = 1, \dots, M \\
\mu_{i,m} &= \alpha_m + \sum_{p=1}^P \nu_{i,m,p}, \quad i = 1, \dots, N, \quad m = 1, \dots, M \\
\nu_{i,m,p} &= \mathbf{F}_{i,p} \Lambda_{p,m}, \quad i = 1, \dots, N, \quad m = 1, \dots, M, \quad p = 1, \dots, P \\
\mathbf{F}_{i,1:P} &\sim \mathcal{N}_P(\gamma, \Omega^{-1}), \quad i = 1, \dots, N \\
\alpha_m &\sim \mathcal{N}(0, 1000), \quad m = 1, \dots, M \\
\gamma_p &= 0, \quad p = 1, \dots, P \\
\Lambda_{p,m} &\sim \mathcal{N}(0, 1000), \quad p = 1, \dots, P, \quad m = 1, \dots, M \\
\Omega &\sim \mathcal{W}(N, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_P \\
\sigma_m &\sim \mathcal{HC}(25), \quad m = 1, \dots, M
\end{aligned}$$

### 27.2. Data

```

data(swiss)
Y <- cbind(swiss$Agriculture, swiss$Examination, swiss$Education,
           swiss$Catholic, swiss$Infant.Mortality)
M <- NCOL(Y) #Number of variables
N <- NROW(Y) #Number of records
P <- trunc(0.5*(2*M + 1 - sqrt(8*M + 1))) #Number of factors to explore
P <- 3 #Number of factors to explore (override for this example)
gamma <- rep(0,P)
S <- diag(P)

```

```

mon.names <- c("LP","mu[1,1]")
parm.names <- parm.names(list(F=matrix(0,N,P), Lambda=matrix(0,P,M),
  Omega=diag(P), alpha=rep(0,M), log.sigma=rep(0,M)),
  uppertri=c(0,0,1,0,0))
MyData <- list(M=M, N=N, P=P, S=S, Y=Y, gamma=gamma, mon.names=mon.names,
  parm.names=parm.names)

```

### 27.3. Initial Values

```

Initial.Values <- c(rep(0,N*P), rep(0,P*M), S[upper.tri(S, diag=TRUE)],
  rep(0,M), rep(0,M))

```

### 27.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[grep("alpha", Data$parm.names)]
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  F <- matrix(parm[grep("F", Data$parm.names)], Data$N, Data$P)
  Lambda <- matrix(parm[grep("Lambda", Data$parm.names)],
    Data$P, Data$M)
  Omega <- matrix(NA, Data$P, Data$P)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[grep("Omega",
    Data$parm.names)]
  Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
  Sigma <- solve(Omega)
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  Omega.prior <- dwishart(Omega, Data$N, Data$S, log=TRUE)
  F.prior <- sum(dmvn(F, Data$gamma, Sigma, log=TRUE))
  Lambda.prior <- sum(dnorm(Lambda, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- Data$Y
  nu <- array(NA, dim=c(Data$N, Data$M, Data$P))
  for (p in 1:Data$P) {nu[, ,p] <- F[,p, drop=FALSE] %*% Lambda[p,]}
  for (m in 1:Data$M) {mu[,m] <- alpha[m] + rowSums(nu[,1,])}
  LL <- sum(dnorm(Data$Y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + sigma.prior + Omega.prior + F.prior +
    Lambda.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,mu[1,1]),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 28. Factor Regression

This example of factor regression is constrained to the case where the number of factors is equal to the number of independent variables (IVs) less the intercept, or  $J - 1$ . The purpose of this form of factor regression is to orthogonalize the IVs with respect to  $\mathbf{y}$ , rather than variable reduction. This method is the combination of confirmatory factor analysis in section 26 and linear regression in section 37.

### 28.1. Form

$$\begin{aligned}
 \mathbf{y}_i &\sim \mathcal{N}(\nu, \sigma_J^2) \\
 \nu &= \mu\beta \\
 \mu_{i,1} &= 1 \\
 \mu_{i,j+1} &= \mu_{i,j}, \quad j = 1, \dots, (J-1) \\
 \mathbf{X}_{i,j} &\sim \mathcal{N}(\mu_{i,j}, \sigma_j^2), \quad i = 1, \dots, N, \quad j = 2, \dots, J \\
 \mu_{i,j} &= \alpha_j + \lambda_j \mathbf{F}_{i,j}, \quad i = 1, \dots, N, \quad j = 2, \dots, J \\
 \mathbf{F}_{i,1:J} &\sim \mathcal{N}_{J-1}(0, \Omega^{-1}), \quad i = 1, \dots, N \\
 \alpha_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1) \\
 \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\
 \lambda_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1) \\
 \sigma_j &\sim \mathcal{HC}(25), \quad j = 1, \dots, J \\
 \Omega &\sim \mathcal{W}(N, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_{J-1}
 \end{aligned}$$

### 28.2. Data

```

data(demonsnacks)
N <- NROW(demonsnacks)
J <- 4
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(7,8,10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
S <- diag((J-1))
mon.names <- "LP"
parm.names <- parm.names(list(alpha=rep(0,J-1), beta=rep(0,J),
  lambda=rep(0,J-1), log.sigma=rep(0,J), F=matrix(0,N,J-1),
  Omega=diag(J-1)), uppertri=c(0,0,0,0,0,1))
MyData <- list(J=J, N=N, S=S, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

### 28.3. Initial Values

```
Initial.Values <- c(rep(0,J-1), rep(0,J), rep(0,J-1), rep(0,J),
  rep(0,N*(J-1)), S[upper.tri(S, diag=TRUE)])
```

### 28.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[grepl("alpha", Data$parm.names)]
  beta <- parm[grepl("beta", Data$parm.names)]
  lambda <- parm[grepl("lambda", Data$parm.names)]
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  F <- matrix(parm[grepl("F", Data$parm.names)], Data$N, Data$J-1)
  Omega <- matrix(NA, Data$J-1, Data$J-1)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[grepl("Omega",
    Data$parm.names)]
  Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
  Sigma <- solve(Omega)
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  lambda.prior <- sum(dnorm(lambda, 0, sqrt(1000), log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  Omega.prior <- dwishart(Omega, Data$N, Data$S, log=TRUE)
  F.prior <- sum(dmvn(F, rep(0,Data$J-1), Sigma, log=TRUE))
  ### Log-Likelihood
  mu <- matrix(alpha, Data$N, Data$J-1, byrow=TRUE) +
    matrix(lambda, Data$N, Data$J-1, byrow=TRUE) * F
  nu <- tcrossprod(beta, cbind(rep(1,Data$N),mu))
  LL <- sum(dnorm(Data$y, nu, sigma[Data$J], log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior + lambda.prior + sigma.prior +
    F.prior + Omega.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=nu, parm=parm)
  return(Modelout)
}
```

## 29. Gamma Regression

### 29.1. Form

$$\mathbf{y} \sim \mathcal{G}(\lambda\tau, \tau)$$

$$\lambda = \exp(\mathbf{X}\beta)$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

$$\tau \sim \mathcal{HC}(25)$$

## 29.2. Data

```

N <- 20
J <- 3
X <- matrix(runif(N*J,-2,2),N,J); X[,1] <- 1
beta <- runif(J,-2,2)
y <- as.vector(round(exp(tcrossprod(beta, X)))) + 0.1 #Must be > 0
mon.names <- c("LP","sigma2")
parm.names <- parm.names(list(beta=rep(0,J), log.tau=0))
MyData <- list(J=J, N=N, X=X, mon.names=mon.names, parm.names=parm.names,
               y=y)

```

## 29.3. Initial Values

```
Initial.Values <- c(rep(0,J), 1)
```

## 29.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[grep("beta", Data$parm.names)]
  tau <- exp(parm[grep("log.tau", Data$parm.names)])
  sigma2 <- 1/tau
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  tau.prior <- dhalfcauchy(tau, 25, log=TRUE)
  ### Log-Likelihood
  lambda <- exp(tcrossprod(beta, Data$X))
  LL <- sum(dgamma(Data$y, tau*lambda, tau, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + tau.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma2), yhat=lambda,
                  parm=parm)
  return(Modelout)
}

```

## 30. GARCH(1,1)

### 30.1. Form

$$\begin{aligned}
\mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma_t^2), \quad t = 1, \dots, T \\
\mathbf{y}^{new} &\sim \mathcal{N}(\mu_{T+1}, \sigma_{new}^2) \\
\mu_t &= \alpha + \phi \mathbf{y}_{t-1}, \quad t = 1, \dots, (T+1) \\
\epsilon_t &= \mathbf{y}_t - \mu_t \\
\alpha &\sim \mathcal{N}(0, 1000) \\
\phi &\sim \mathcal{N}(0, 1000) \\
\sigma_{new}^2 &= \theta_1 + \theta_2 \epsilon_T^2 + \theta_3 \sigma_T^2 \\
\sigma_t^2 &= \theta_1 + \theta_2 \epsilon_{t-1}^2 + \theta_3 \sigma_{t-1}^2 \\
\theta_k &= \frac{1}{1 + \exp(-\theta_k)}, \quad k = 1, \dots, 3 \\
\theta_k &\sim \mathcal{N}(0, 1000) \in [-10, 10], \quad k = 1, \dots, 3
\end{aligned}$$

### 30.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
      0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
      0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
      -0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
      0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,

```

```

-0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)
T <- length(y)
mon.names <- c("LP", "ynew", "sigma2.new")
parm.names <- c("alpha", "phi", "logit.theta[1]", "logit.theta[2]",
               "logit.theta[3]")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 30.3. Initial Values

```
Initial.Values <- c(rep(0,2), rep(0,3))
```

### 30.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]; phi <- parm[2]
  theta <- invlogit(interval(parm[grep("logit.theta",
    Data$parm.names)], -10, 10))
  parm[grep("logit.theta", Data$parm.names)] <- logit(theta)
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)
  theta.prior <- sum(dnorm(theta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- c(alpha, alpha + phi*Data$y[-Data$T])
  ynew <- alpha + phi*Data$y[Data$T]
  epsilon <- Data$y - mu
  sigma2 <- c(theta[1], theta[1] + theta[2]*epsilon[-Data$T]^2)
  sigma2[-1] <- sigma2[-1] + theta[3]*sigma2[-Data$T]
  sigma2.new <- theta[1] + theta[2]*epsilon[Data$T]^2 +
    theta[3]*sigma2[Data$T]
  LL <- sum(dnorm(Data$y, mu, sqrt(sigma2), log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + phi.prior + theta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, ynew, sigma2.new),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 31. GARCH-M(1,1)

### 31.1. Form

$$\begin{aligned}
\mathbf{y}_t &\sim \mathcal{N}(\mu_t, \sigma_t^2), \quad t = 1, \dots, T \\
\mathbf{y}^{new} &\sim \mathcal{N}(\mu_{T+1}, \sigma_{new}^2) \\
\mu_t &= \alpha + \phi \mathbf{y}_{t-1} + \delta \sigma_{t-1}^2, \quad t = 1, \dots, (T+1) \\
\epsilon_t &= \mathbf{y}_t - \mu_t \\
\alpha &\sim \mathcal{N}(0, 1000) \\
\phi &\sim \mathcal{N}(0, 1000) \\
\sigma_{new}^2 &= \theta_1 + \theta_2 \epsilon_T^2 + \theta_3 \sigma_T^2 \\
\sigma_t^2 &= \theta_1 + \theta_2 \epsilon_{t-1}^2 + \theta_3 \sigma_{t-1}^2 \\
\theta_k &= \frac{1}{1 + \exp(-\theta_k)}, \quad k = 1, \dots, 3 \\
\theta_k &\sim \mathcal{N}(0, 1000) \in [-10, 10], \quad k = 1, \dots, 3
\end{aligned}$$

### 31.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
      2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
      1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
      -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
      0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
      1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
      0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
      0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,
      0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
      -0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
      0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
      -0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
      -0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
      -0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
      0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
      -0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
      0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
      -0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
      -0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
      0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
      -0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
      0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
      0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
      0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
      -0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
      0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,

```

```

-0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)
T <- length(y)
mon.names <- c("LP", "ynew", "sigma2.new")
parm.names <- c("alpha", "phi", "delta", "logit.theta[1]", "logit.theta[2]",
               "logit.theta[3]")
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 31.3. Initial Values

```
Initial.Values <- c(rep(0,3), rep(0,3))
```

### 31.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1]; phi <- parm[2]; delta <- parm[3]
  theta <- invlogit(interval(parm[grep("logit.theta",
    Data$parm.names)], -10, 10))
  parm[grep("logit.theta", Data$parm.names)] <- logit(theta)
  ### Log(Prior Densities)
  alpha.prior <- dnorm(alpha, 0, sqrt(1000), log=TRUE)
  phi.prior <- dnorm(phi, 0, sqrt(1000), log=TRUE)
  delta.prior <- dnorm(delta, 0, sqrt(1000), log=TRUE)
  theta.prior <- sum(dnorm(theta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- c(alpha, alpha + phi*Data$y[-Data$T])
  epsilon <- Data$y - mu
  sigma2 <- c(theta[1], theta[1] + theta[2]*epsilon[-Data$T]^2)
  sigma2[-1] <- sigma2[-1] + theta[3]*sigma2[-Data$T]
  sigma2.new <- theta[1] + theta[2]*epsilon[Data$T]^2 +
    theta[3]*sigma2[Data$T]
  mu <- mu + delta*sigma2
  ynew <- alpha + phi*Data$y[Data$T] + delta*sigma2[Data$T]
  LL <- sum(dnorm(Data$y, mu, sqrt(sigma2), log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + phi.prior + delta.prior + theta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, ynew, sigma2.new),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 32. Geographically Weighted Regression

### 32.1. Form

$$\begin{aligned}
\mathbf{y}_{i,k} &\sim \mathcal{N}(\mu_{i,k}, \tau_{i,k}^{-1}), \quad i = 1, \dots, N, \quad k = 1, \dots, N \\
\mu_{i,1:N} &= \mathbf{X}\beta_{i,1:J} \\
\tau &= \frac{1}{\sigma^2} \mathbf{w}\nu \\
\mathbf{w} &= \frac{\exp(-0.5\mathbf{Z}^2)}{\mathbf{h}} \\
\alpha &\sim \mathcal{U}(1.5, 100) \\
\beta_{i,j} &\sim \mathcal{N}(0, 1000), \quad i = 1, \dots, N, \quad j = 1, \dots, J \\
\mathbf{h} &\sim \mathcal{N}(0.1, 1000) \in [0.1, \infty] \\
\nu_{i,k} &\sim \mathcal{G}(\alpha, 2), \quad i = 1, \dots, N, \quad k = 1, \dots, N \\
\sigma_i &\sim \mathcal{HC}(25), \quad i = 1, \dots, N
\end{aligned}$$

### 32.2. Data

```

crime <- c(18.802, 32.388, 38.426, 0.178, 15.726, 30.627, 50.732,
          26.067, 48.585, 34.001, 36.869, 20.049, 19.146, 18.905, 27.823,
          16.241, 0.224, 30.516, 33.705, 40.970, 52.794, 41.968, 39.175,
          53.711, 25.962, 22.541, 26.645, 29.028, 36.664, 42.445, 56.920,
          61.299, 60.750, 68.892, 38.298, 54.839, 56.706, 62.275, 46.716,
          57.066, 54.522, 43.962, 40.074, 23.974, 17.677, 14.306, 19.101,
          16.531, 16.492)
income <- c(21.232, 4.477, 11.337, 8.438, 19.531, 15.956, 11.252,
           16.029, 9.873, 13.598, 9.798, 21.155, 18.942, 22.207, 18.950,
           29.833, 31.070, 17.586, 11.709, 8.085, 10.822, 9.918, 12.814,
           11.107, 16.961, 18.796, 11.813, 14.135, 13.380, 17.017, 7.856,
           8.461, 8.681, 13.906, 14.236, 7.625, 10.048, 7.467, 9.549,
           9.963, 11.618, 13.185, 10.655, 14.948, 16.940, 18.739, 18.477,
           18.324, 25.873)
housing <- c(44.567, 33.200, 37.125, 75.000, 80.467, 26.350, 23.225,
            28.750, 18.000, 96.400, 41.750, 47.733, 40.300, 42.100, 42.500,
            61.950, 81.267, 52.600, 30.450, 20.300, 34.100, 23.600, 27.000,
            22.700, 33.500, 35.800, 26.800, 27.733, 25.700, 43.300, 22.850,
            17.900, 32.500, 22.500, 53.200, 18.800, 19.900, 19.700, 41.700,
            42.900, 30.600, 60.000, 19.975, 28.450, 31.800, 36.300, 39.600,
            76.100, 44.333)
easting <- c(35.62, 36.50, 36.71, 33.36, 38.80, 39.82, 40.01, 43.75,
            39.61, 47.61, 48.58, 49.61, 50.11, 51.24, 50.89, 48.44, 46.73,
            43.44, 43.37, 41.13, 43.95, 44.10, 43.70, 41.04, 43.23, 42.67,
            41.21, 39.32, 41.09, 38.3, 41.31, 39.36, 39.72, 38.29, 36.60,
            37.60, 37.13, 37.85, 35.95, 35.72, 35.76, 36.15, 34.08, 30.32,
            27.94, 27.27, 24.25, 25.47, 29.02)

```

```

northing <- c(42.38, 40.52, 38.71, 38.41, 44.07, 41.18, 38.00, 39.28,
             34.91, 36.42, 34.46, 32.65, 29.91, 27.80, 25.24, 27.93, 31.91,
             35.92, 33.46, 33.14, 31.61, 30.40, 29.18, 28.78, 27.31, 24.96,
             25.90, 25.85, 27.49, 28.82, 30.90, 32.88, 30.64, 30.35, 32.09,
             34.08, 36.12, 36.30, 36.40, 35.60, 34.66, 33.92, 30.42, 28.26,
             29.85, 28.21, 26.69, 25.71, 26.58)
N <- length(crime)
J <- 3 #Number of predictors, including the intercept
X <- matrix(c(rep(1,N), income, housing),N,J)
D <- as.matrix(dist(cbind(northing,easting), diag=TRUE, upper=TRUE))
Z <- D / sd(as.vector(D))
y <- matrix(0,N,N); for (i in 1:N) {for (k in 1:N) {y[i,k] <- crime[k]}}
mon.names <- c("LP",parm.names(list(LAR2=rep(0,N))))
parm.names <- parm.names(list(alpha=0, beta=matrix(0,N,J), log.h=0,
             log.nu=matrix(0,N,N), log.sigma=rep(0,N)))
MyData <- list(J=J, N=N, X=X, Z=Z, latitude=northing, longitude=easting,
             mon.names=mon.names, parm.names=parm.names, y=y)

```

### 32.3. Initial Values

```

Initial.Values <- c(runif(1,1.5,100), rep(0,N*J), log(1), rep(0,N*N),
             log(rep(1,N)))

```

### 32.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- interval(parm[grepl("alpha", Data$parm.names)], 1.5, 100)
  parm[grepl("alpha", Data$parm.names)] <- alpha
  beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$N, Data$J)
  h <- exp(parm[grepl("log.h", Data$parm.names)]) + 0.1
  nu <- exp(matrix(parm[grepl("log.nu", Data$parm.names)],
             Data$N, Data$N))
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  alpha.prior <- dunif(alpha, 1.5, 100, log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  h.prior <- dtrunc(h, "norm", a=0.1, b=Inf, mean=0.1, sd=sqrt(1000),
             log=TRUE)
  nu.prior <- sum(dgamma(nu, alpha, 2, log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  w <- exp(-0.5 * Z^2) / h
  tau <- (1/sigma^2) * w * nu
  mu <- matrix(NA, Data$N, Data$N)
}

```

```

for (i in 1:N) {mu[i,] <- tcrossprod(beta[i,], Data$X)}
LL <- sum(dnorm(Data$y, mu, sqrt(1/tau), log=TRUE))
WSE <- w * nu * (Data$y - mu)^2; w.y <- w * nu * Data$y
WMSE <- rowMeans(WSE); y.w <- rowSums(w.y) / rowSums(w)
LAR2 <- 1 - WMSE / sd(y.w)^2
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + h.prior + nu.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,LAR2), yhat=mu,
  parm=parm)
return(Modelout)
}

```

### 33. Inverse Gaussian Regression

#### 33.1. Form

$$\begin{aligned}
 \mathbf{y} &\sim \mathcal{N}^{-1}(\mu, \lambda) \\
 \mu &= \exp(\mathbf{X}\beta) + C \\
 \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\
 \lambda &\sim \mathcal{HC}(25)
 \end{aligned}$$

where  $C$  is a small constant, such as 1.0E-10.

#### 33.2. Data

```

N <- 100
J <- 3 #Number of predictors, including the intercept
X <- matrix(1,N,J)
for (j in 2:J) {X[,j] <- rnorm(N,runif(1,-3,3),runif(1,0.1,1))}
beta.orig <- runif(J,-3,3)
e <- rnorm(N,0,0.1)
y <- exp(as.vector(tcrossprod(beta.orig, X)) + e)
mon.names <- c("LP","lambda")
parm.names <- parm.names(list(beta=rep(0,J), log.lambda=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

#### 33.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

#### 33.4. Model

```
Model <- function(parm, Data)
```

```

{
### Parameters
beta <- parm[1:Data$J]
lambda <- exp(parm[Data$J+1])
### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
lambda.prior <- dhalfcauchy(lambda, 25, log=TRUE)
### Log-Likelihood
mu <- exp(tcrossprod(beta, Data$X)) + 1.0E-10
LL <- sum(dinvgaussian(Data$y, mu, lambda, log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + lambda.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,lambda), yhat=mu,
  parm=parm)
return(Modelout)
}

```

## 34. Kriging

This is an example of universal kriging of  $\mathbf{y}$  given  $\mathbf{X}$ , regression effects  $\beta$ , and spatial effects  $\zeta$ . Euclidean distance between spatial coordinates (longitude and latitude) is used for each of  $i = 1, \dots, N$  records of  $\mathbf{y}$ . An additional record is created from the same data-generating process to compare the accuracy of interpolation. For the spatial component,  $\phi$  is the rate of spatial decay and  $\kappa$  is the scale.  $\kappa$  is often difficult to identify, so it is set to 1 (Gaussian), but may be allowed to vary up to 2 (Exponential). In practice,  $\phi$  is also often difficult to identify. While  $\Sigma$  is spatial covariance, spatial correlation is  $\rho = \exp(-\phi \mathbf{D})$ . To extend this to a large data set, consider the predictive process kriging example in section 35.

### 34.1. Form

$$\begin{aligned}
\mathbf{y} &\sim \mathcal{N}(\mu, \sigma_1^2) \\
\mu &= \mathbf{X}\beta + \zeta \\
\mathbf{y}^{new} &= \mathbf{X}\beta + \sum_{i=1}^N \left( \frac{\rho_i}{\sum \rho} \zeta_i \right) \\
\rho &= \exp(-\phi \mathbf{D}^{new})^\kappa \\
\zeta &\sim \mathcal{N}_N(\zeta_\mu, \Sigma) \\
\Sigma &= \sigma_2^2 \exp(-\phi \mathbf{D})^\kappa \\
\beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 2 \\
\sigma_j &\sim \mathcal{HC}(25), \quad j = 1, \dots, 2 \\
\phi &\sim \mathcal{U}(1, 5)
\end{aligned}$$

$$\zeta_\mu = 0$$

$$\kappa = 1$$

### 34.2. Data

```

N <- 20
longitude <- runif(N+1,0,100)
latitude <- runif(N+1,0,100)
D <- as.matrix(dist(cbind(longitude,latitude), diag=TRUE, upper=TRUE))
Sigma <- 10000 * exp(-1.5 * D)
zeta <- as.vector(apply(rmvn(1000, rep(0,N+1), Sigma), 2, mean))
beta <- c(50,2)
X <- matrix(runif((N+1)*2,-2,2),(N+1),2); X[,1] <- 1
mu <- as.vector(tcrossprod(beta, X))
y <- mu + zeta
longitude.new <- longitude[N+1]; latitude.new <- latitude[N+1]
Xnew <- X[N+1,]; ynew <- y[N+1]
longitude <- longitude[1:N]; latitude <- latitude[1:N]
X <- X[1:N,]; y <- y[1:N]
D <- as.matrix(dist(cbind(longitude,latitude), diag=TRUE, upper=TRUE))
D.new <- sqrt((longitude - longitude.new)^2 + (latitude - latitude.new)^2)
mon.names <- c("LP","sigma[1]","sigma[2]","ynew")
parm.names <- parm.names(list(zeta=rep(0,N), beta=rep(0,2),
  log.sigma=rep(0,2), phi=0))
MyData <- list(D=D, D.new=D.new, N=N, X=X, Xnew=Xnew, latitude=latitude,
  longitude=longitude, mon.names=mon.names, parm.names=parm.names,
  y=y)

```

### 34.3. Initial Values

```
Initial.Values <- c(rep(0,N), rep(0,2), rep(0,2), 1)
```

### 34.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[grep("beta", Data$parm.names)]
  zeta <- parm[grep("zeta", Data$parm.names)]
  kappa <- 1
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  phi <- interval(parm[grep("phi", Data$parm.names)], 1, 5)
  parm[grep("phi", Data$parm.names)] <- phi
  Sigma <- sigma[2]*sigma[2] * exp(-phi * Data$D)^kappa
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  zeta.prior <- dmvn(zeta, rep(0, Data$N), Sigma, log=TRUE)
}

```

```

sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
phi.prior <- dunif(phi, 1, 5, log=TRUE)
### Interpolation
rho <- exp(-phi * Data$D.new)^kappa
ynew <- sum(beta * Data$Xnew) + sum(rho / sum(rho) * zeta)
### Log-Likelihood
mu <- tcrossprod(beta, Data$X) + zeta
LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + zeta.prior + sigma.prior + phi.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,ynew),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 35. Kriging, Predictive Process

The first  $K$  of  $N$  records in  $\mathbf{y}$  are used as knots for the parent process, and the predictive process involves records  $(K+1), \dots, N$ . For more information on kriging, see section 34.

### 35.1. Form

$$\mathbf{y} \sim \mathcal{N}(\mu, \sigma_1^2)$$

$$\mu_{1:K} = \mathbf{X}_{1:K,1:J}\beta + \zeta$$

$$\mu_{(K+1):N} = \mathbf{X}_{(K+1):N,1:J}\beta + \sum_{p=1}^{N-K} \frac{\lambda_{p,1:K}}{\sum_{q=1}^{N-K} \lambda_{q,1:K}} \zeta^T$$

$$\lambda = \exp(-\phi \mathbf{D}_P)^\kappa$$

$$\mathbf{y}^{new} = \mathbf{X}\beta + \sum_{k=1}^K \left( \frac{\rho_k}{\sum \rho} \zeta_k \right)$$

$$\rho = \exp(-\phi \mathbf{D}^{new})^\kappa$$

$$\zeta \sim \mathcal{N}_K(0, \Sigma)$$

$$\Sigma = \sigma_2^2 \exp(-\phi \mathbf{D})^\kappa$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 2$$

$$\sigma_j \sim \mathcal{HC}(25), \quad j = 1, \dots, 2$$

$$\phi \sim \mathcal{N}(0, 1000) \in [0, \infty]$$

$$\kappa = 1$$

### 35.2. Data

```

N <- 100
K <- 30 #Number of knots
longitude <- runif(N+1,0,100)
latitude <- runif(N+1,0,100)
D <- as.matrix(dist(cbind(longitude,latitude), diag=TRUE, upper=TRUE))
Sigma <- 10000 * exp(-1.5 * D)
zeta <- as.vector(apply(rmvn(1000, rep(0,N+1), Sigma), 2, mean))
beta <- c(50,2)
X <- matrix(runif((N+1)*2,-2,2),(N+1),2); X[,1] <- 1
mu <- as.vector(tcrossprod(beta, X))
y <- mu + zeta
longitude.new <- longitude[N+1]; latitude.new <- latitude[N+1]
Xnew <- X[N+1,]; ynew <- y[N+1]
longitude <- longitude[1:N]; latitude <- latitude[1:N]
X <- X[1:N,]; y <- y[1:N]
D <- as.matrix(dist(cbind(longitude[1:K],latitude[1:K]), diag=TRUE,
  upper=TRUE))
D.P <- matrix(0, N-K, K)
for (i in (K+1):N) {
  D.P[K+1-i,] <- sqrt((longitude[1:K] - longitude[i])^2 +
    (latitude[1:K] - latitude[i])^2)}
D.new <- sqrt((longitude[1:K] - longitude.new)^2 +
  (latitude[1:K] - latitude.new)^2)
mon.names <- c("LP","sigma[1]","sigma[2]","ynew")
parm.names <- parm.names(list(zeta=rep(0,K), beta=rep(0,2),
  log.sigma=rep(0,2), log.phi=0))
MyData <- list(D=D, D.new=D.new, D.P=D.P, K=K, N=N, X=X, Xnew=Xnew,
  latitude=latitude, longitude=longitude,
  mon.names=mon.names, parm.names=parm.names, y=y)

```

### 35.3. Initial Values

```
Initial.Values <- c(rep(0,K), c(mean(y), 0), rep(0,2), log(1))
```

### 35.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[grep("beta", Data$parm.names)]
  zeta <- parm[grep("zeta", Data$parm.names)]
  kappa <- 1
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  phi <- exp(parm[grep("log.phi", Data$parm.names)])
  Sigma <- sigma[2]*sigma[2] * exp(-phi * Data$D)^kappa

```

```

### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
zeta.prior <- dmvn(zeta, rep(0, Data$K), Sigma, log=TRUE)
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
phi.prior <- dunif(phi, 1, 5, log=TRUE)
### Interpolation
rho <- exp(-phi * Data$D.new)^kappa
ynew <- sum(beta * Data$Xnew) + sum(rho / sum(rho) * zeta)
### Log-Likelihood
mu <- tcrossprod(beta, Data$X)
mu[1:Data$K] <- mu[1:Data$K] + zeta
lambda <- exp(-phi * Data$D.P)^kappa
mu[(Data$K+1):Data$N] <- mu[(Data$K+1):Data$N] +
  rowSums(lambda / rowSums(lambda) *
    matrix(zeta, Data$N - Data$K, Data$K, byrow=TRUE))
LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + zeta.prior + sigma.prior + phi.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,ynew),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 36. Laplace Regression

This linear regression specifies that  $\mathbf{y}$  is Laplace-distributed, where it is usually Gaussian or normally-distributed. It has been claimed that it should be surprising that the normal distribution became the standard, when the Laplace distribution usually fits better and has wider tails (Kotz, Kozubowski, and Podgorski 2001). Another popular alternative is to use the t-distribution (see Robust Regression in section 58), though it is more computationally expensive to estimate, because it has three parameters. The Laplace distribution has only two parameters, location and scale like the normal distribution, and is computationally easier to fit. This example could be taken one step further, and the parameter vector  $\beta$  could be Laplace-distributed. Laplace's Demon recommends that users experiment with replacing the normal distribution with the Laplace distribution.

### 36.1. Form

$$\mathbf{y} \sim \mathcal{L}(\mu, \sigma^2)$$

$$\mu = \mathbf{X}\beta$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

$$\sigma \sim \mathcal{HC}(25)$$

### 36.2. Data

```

N <- 10000
J <- 5
X <- matrix(1,N,J)
for (j in 2:J) {X[,j] <- rnorm(N,runif(1,-3,3),runif(1,0.1,1))}
beta <- runif(J,-3,3)
e <- rlaplace(N,0,0.1)
y <- as.vector(tcrossprod(beta, X) + e)
mon.names <- c("LP", "sigma")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 36.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 36.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(dlaplace(Data$y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, sigma), yhat=mu,
    parm=parm)
  return(Modelout)
}

```

## 37. Linear Regression

### 37.1. Form

$$\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \sigma^2)$$

$$\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

$$\sigma \sim \mathcal{HC}(25)$$

### 37.2. Data

```

N <- 10000
J <- 5
X <- matrix(1,N,J)
for (j in 2:J) {X[,j] <- rnorm(N,runif(1,-3,3),runif(1,0.1,1))}
beta <- runif(J,-3,3)
e <- rnorm(N,0,0.1)
y <- as.vector(tcrossprod(beta, X) + e)
mon.names <- c("LP", "sigma")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 37.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 37.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dgamma(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, sigma), yhat=mu,
    parm=parm)
  return(Modelout)
}

```

## 38. Linear Regression, Frequentist

By eliminating prior probabilities, a frequentist linear regression example is presented. Although frequentism is not endorsed here, the purpose of this example is to illustrate how the **LaplaceDemon** package can be used for Bayesian or frequentist inference.

### 38.1. Form

$$\mathbf{y} \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = \mathbf{X}\beta$$

### 38.2. Data

```

N <- 10000
J <- 5
X <- matrix(1,N,J)
for (j in 2:J) {X[,j] <- rnorm(N,runif(1,-3,3),runif(1,0.1,1))}
beta <- runif(J,-3,3)
e <- rnorm(N,0,0.1)
y <- as.vector(tcrossprod(beta, X) + e)
mon.names <- c("LL", "sigma")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 38.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 38.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
  Modelout <- list(LP=LL, Dev=-2*LL, Monitor=c(LL, sigma), yhat=mu,
    parm=parm)
  return(Modelout)
}

```

## 39. Linear Regression, Multilevel

### 39.1. Form

$$\mathbf{y} \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu_i = \mathbf{X}\beta_{\mathbf{m}[i],1:J}$$

$$\begin{aligned}\beta_{g,1:J} &\sim \mathcal{N}_J(\gamma, \Sigma), \quad g = 1, \dots, G \\ \Sigma &= \Omega^{-1} \\ \Omega &\sim \mathcal{W}(J, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_J \\ \gamma_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \sigma &\sim \mathcal{HC}(25)\end{aligned}$$

where  $\mathbf{m}$  is a vector of length  $N$ , and each element indicates the multilevel group ( $g = 1, \dots, G$ ) for the associated record.

### 39.2. Data

```
N <- 30
J <- 2 ### Number of predictors (including intercept)
G <- 2 ### Number of Multilevel Groups
X <- matrix(rnorm(N,0,1),N,J); X[,1] <- 1
Sigma <- matrix(runif(J*J,-1,1),J,J)
diag(Sigma) <- runif(J,1,5)
gamma <- runif(J,-1,1)
beta <- matrix(NA,G,J)
for (g in 1:G) {beta[g,] <- rmvn(1, gamma, Sigma)}
m <- round(runif(N,0.5,(G+0.49))) ### Multilevel group indicator
y <- rowSums(beta[m,] * X) + rnorm(N,0,0.1)
S <- diag(J)
mon.names <- c("LP","sigma")
parm.names <- parm.names(list(beta=matrix(0,G,J), log.sigma=0,
  gamma=rep(0,J), Omega=S), uppertri=c(0,0,0,1))
MyData <- list(G=G, J=J, N=N, S=S, X=X, m=m, mon.names=mon.names,
  parm.names=parm.names, y=y)
```

### 39.3. Initial.Values

```
Initial.Values <- c(rep(0,G*J), log(1), rep(0,J),
  S[upper.tri(S, diag=TRUE)])
```

### 39.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- matrix(parm[1:(Data$G * Data$J)], Data$G, Data$J)
  gamma <- parm[grepl("gamma", Data$parm.names)]
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  Omega <- matrix(NA, Data$J, Data$J)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[min(grepl("Omega",
    Data$parm.names)): max(grepl("Omega", Data$parm.names))]
```

```

Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
Sigma <- solve(Omega)
### Log(Prior Densities)
Omega.prior <- dwishart(Omega, Data$J, Data$S, log=TRUE)
beta.prior <- sum(dmvn(beta, gamma, Sigma, log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sqrt(100), log=TRUE))
sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
### Log-Likelihood
mu <- rowSums(beta[Data$m,] * Data$X)
LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
### Log-Posterior
LP <- LL + Omega.prior + beta.prior + gamma.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 40. Linear Regression with Full Missingness

With ‘full missingness’, there are missing values for both the response and at least one predictor. This is a minimal example, since there are missing values in only one of the predictors. Initial values do not need to be specified for missing values in a predictor, unless another predictor variable with missing values is used to predict the missing values of a predictor. More effort is involved in specifying a model with a missing predictor that is predicted by another missing predictor. The full likelihood approach to full missingness is excellent as long as the model is identifiable. When it is not identifiable, then imputation may be done in a previous stage. In this example,  $\mathbf{X}[,2]$  is the only predictor with missing values.

### 40.1. Form

$$\begin{aligned}
 \mathbf{y} &\sim \mathcal{N}(\mu_2, \sigma_2^2) \\
 \mu_2 &= \mathbf{X}\beta \\
 \mathbf{X}_{1:N,2} &\sim \mathcal{N}(\mu_1, \sigma_1^2) \\
 \mu_1 &= \mathbf{X}_{1:N,(1,3:J)}\alpha \\
 \alpha_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1) \\
 \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\
 \sigma_k &\sim \mathcal{HC}(25), \quad k = 1, \dots, 2
 \end{aligned}$$

### 40.2. Data

```

N <- 1000
J <- 5

```

```

X <- matrix(runif(N*J,-2,2),N,J)
X[,1] <- 1
alpha <- runif((J-1),-2,2)
X[,2] <- tcrossprod(alpha, X[, -2]) + rnorm(N,0,0.1)
beta <- runif(J,-2,2)
y <- as.vector(tcrossprod(beta, X) + rnorm(N,0,0.1))
y[sample(1:N, round(N*0.05))] <- NA
M <- ifelse(is.na(y), 1, 0)
X[sample(1:N, round(N*0.05)),2] <- NA
mon.names <- c("LP","sigma[1]","sigma[2]")
parm.names <- parm.names(list(alpha=rep(0,J-1), beta=rep(0,J),
  log.sigma=rep(0,2)))
MyData <- list(J=J, M=M, N=N, X=X, mon.names=mon.names, parm.names=parm.names,
  y=y)

```

### 40.3. Initial Values

```
Initial.Values <- c(rep(0,(J-1)), rep(0,J), rep(0,2))
```

### 40.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1:(Data$J-1)]
  beta <- parm[Data$J:(2*Data$J - 1)]
  sigma <- exp(parm[(2*Data$J):(2*Data$J+1)])
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  mu1 <- tcrossprod(alpha, Data$X[, -2])
  X.imputed <- Data$X
  X.imputed[,2] <- ifelse(is.na(Data$X[,2]), mu1, Data$X[,2])
  LL1 <- sum(dnorm(X.imputed[,2], mu1, sigma[1], log=TRUE))
  mu2 <- tcrossprod(beta, X.imputed)
  y.imputed <- ifelse(is.na(Data$y), mu2, Data$y)
  LL2 <- sum((1-Data$M) * dnorm(y.imputed, mu2, sigma[2], log=TRUE))
  ### Log-Posterior
  LP <- LL1 + LL2 + alpha.prior + beta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL2, Monitor=c(LP,sigma),
    yhat=mu2, parm=parm)
  return(Modelout)
}

```

## 41. Linear Regression with Missing Response

Initial values do not need to be specified for missing values in this response,  $\mathbf{y}$ . Instead, at each iteration, missing values in  $\mathbf{y}$  are replaced with their estimate in  $\mu$ .

### 41.1. Form

$$\begin{aligned}\mathbf{y} &\sim \mathcal{N}(\mu, \sigma^2) \\ \mu &= \mathbf{X}\beta \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \sigma &\sim \mathcal{HC}(25)\end{aligned}$$

### 41.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
y[sample(1:N, round(N*0.05))] <- NA
M <- ifelse(is.na(y), 1, 0)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- c("LP", "sigma")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, M=M, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

### 41.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 41.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dgamma(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  y.imputed <- ifelse(is.na(Data$y), mu, Data$y)
  LL <- sum((1-Data$M) * dnorm(y.imputed, mu, sigma, log=TRUE))
}
```

```

### Log-Posterior
LP <- LL + beta.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 42. MANCOVA

Since this is a multivariate extension of ANCOVA, please see the ANCOVA example in section 1 for a univariate introduction.

### 42.1. Form

$$\begin{aligned}
 \mathbf{Y}_{i,1:J} &\sim \mathcal{N}(\mu_{i,1:J}, \Sigma), \quad i = 1, \dots, N \\
 \mu_{i,k} &= \alpha_k + \beta_{k,\mathbf{X}[i,1]} + \gamma_{k,\mathbf{X}[i,1]} + \mathbf{X}_{1:N,3:(C+J)} \delta_{k,1:C} \\
 \epsilon_{i,k} &= \mathbf{Y}_{i,k} - \mu_{i,k} \\
 \alpha_k &\sim \mathcal{N}(0, 1000), \quad k = 1, \dots, K \\
 \beta_{k,l} &\sim \mathcal{N}(0, \sigma_1^2), \quad l = 1, \dots, (L-1) \\
 \beta_{1:K,L} &= - \sum_{l=1}^{L-1} \beta_{1:K,l} \\
 \gamma_{k,m} &\sim \mathcal{N}(0, \sigma_2^2), \quad m = 1, \dots, (M-1) \\
 \gamma_{1:K,M} &= - \sum_{m=1}^{M-1} \beta_{1:K,m} \\
 \delta_{k,c} &\sim \mathcal{N}(0, 1000) \\
 \Omega &\sim \mathcal{W}(N, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_K \\
 \Sigma &= \Omega^{-1} \\
 \sigma_{1:J} &\sim \mathcal{HC}(25)
 \end{aligned}$$

### 42.2. Data

```

C <- 2 #Number of covariates
J <- 2 #Number of factors (treatments)
K <- 3 #Number of endogenous (dependent) variables
L <- 4 #Number of levels in factor (treatment) 1
M <- 5 #Number of levels in factor (treatment) 2
N <- 100
X <- matrix(cbind(round(runif(N, 0.5, L+0.49)), round(runif(N, 0.5, M+0.49))),
  runif(C*N, 0, 1)), N, J + C)

```

```

alpha <- runif(K,-1,1)
beta <- matrix(runif(K*L,-2,2), K, L)
beta[,L] <- -rowSums(beta[, -L])
gamma <- matrix(runif(K*M,-2,2), K, M)
gamma[,M] <- -rowSums(gamma[, -M])
delta <- matrix(runif(K*C), K, C)
Y <- matrix(NA,N,K)
for (k in 1:K) {
  Y[,k] <- alpha[k] + beta[k,X[,1]] + gamma[k,X[,2]] +
    tcrossprod(delta[k,], X[, -c(1,2)]) + rnorm(1,0,0.1)}
S <- diag(K)
mon.names <- c("LP", "s.o.beta", "s.o.gamma", "s.o.epsilon",
  parm.names(list(s.beta=rep(0,K), s.gamma=rep(0,K),
    s.epsilon=rep(0,K))))
parm.names <- parm.names(list(alpha=rep(0,K), beta=matrix(0,K,(L-1)),
  gamma=matrix(0,K,(M-1)), delta=matrix(0,K,C), Omega=diag(K),
  log.sigma=rep(0,2)), uppertri=c(0,0,0,0,1,0))
MyData <- list(C=C, J=J, K=K, L=L, M=M, N=N, S=S, X=X, Y=Y,
  mon.names=mon.names, parm.names=parm.names)

```

### 42.3. Initial Values

```

Initial.Values <- c(rep(0,K), rep(0,K*(L-1)), rep(0,K*(M-1)),
  rep(0,C*K), S[upper.tri(S, diag=TRUE)], rep(0,2))

```

### 42.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[grepl("alpha", Data$parm.names)]
  beta <- matrix(c(parm[grepl("beta", Data$parm.names)]), rep(0,K)),
    Data$K, Data$L)
  beta[,L] <- -rowSums(beta[, -L])
  gamma <- matrix(c(parm[grepl("gamma", Data$parm.names)]), rep(0,K)),
    Data$K, Data$M)
  gamma[,M] <- -rowSums(gamma[, -M])
  delta <- matrix(parm[grepl("delta", Data$parm.names)], Data$K, Data$C)
  Omega <- matrix(NA, Data$K, Data$K)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[grepl("Omega",
    Data$parm.names)]
  Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
  Sigma <- solve(Omega)
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))

```

```

beta.prior <- sum(dnorm(beta, 0, sigma[1], log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sigma[2], log=TRUE))
delta.prior <- sum(dnorm(delta, 0, sqrt(1000), log=TRUE))
Omega.prior <- dwishart(Omega, NROW(Data$S), Data$S, log=TRUE)
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
mu <- matrix(0, Data$N, Data$K)
for (k in 1:K) {
  mu[,k] <- alpha[k] + beta[k, Data$X[,1]] + gamma[k, Data$X[,2]] +
    tcrossprod(delta[k,], Data$X[, -c(1,2)])}
LL <- sum(dmvn(Data$Y, mu, Sigma, log=TRUE))
### Variance Components, Omnibus
s.o.beta <- sd(as.vector(beta))
s.o.gamma <- sd(as.vector(gamma))
s.o.epsilon <- sd(as.vector(Data$Y - mu))
### Variance Components, Univariate
s.beta <- sd(t(beta))
s.gamma <- sd(t(gamma))
s.epsilon <- sd(Data$Y - mu)
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + gamma.prior + delta.prior +
  Omega.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, s.o.beta, s.o.gamma,
  s.o.epsilon, s.beta, s.gamma, s.epsilon), yhat=mu, parm=parm)
return(Modelout)
}

```

## 43. MANOVA

Since this is a multivariate extension of ANOVA, please see the two-way ANOVA example in section 3 for a univariate introduction.

### 43.1. Form

$$\mathbf{Y}_{i,1:J} \sim \mathcal{N}(\mu_{i,1:J}, \Sigma), \quad i = 1, \dots, N$$

$$\mu_{i,k} = \alpha_k + \beta_{k,\mathbf{X}[i,1]} + \gamma_{k,\mathbf{X}[i,1]}$$

$$\epsilon_{i,k} = \mathbf{Y}_{i,k} - \mu_{i,k}$$

$$\alpha_k \sim \mathcal{N}(0, 1000), \quad k = 1, \dots, K$$

$$\beta_{k,l} \sim \mathcal{N}(0, \sigma_1^2), \quad l = 1, \dots, (L-1)$$

$$\beta_{1:K,L} = - \sum_{l=1}^{L-1} \beta_{1:K,l}$$

$$\begin{aligned}\gamma_{k,m} &\sim \mathcal{N}(0, \sigma_2^2), \quad m = 1, \dots, (M-1) \\ \gamma_{1:K,M} &= - \sum_{m=1}^{M-1} \beta_{1:K,m} \\ \Omega &\sim \mathcal{W}(N, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_K \\ \Sigma &= \Omega^{-1} \\ \sigma_{1:J} &\sim \mathcal{HC}(25)\end{aligned}$$

### 43.2. Data

```
J <- 2 #Number of factors (treatments)
K <- 3 #Number of endogenous (dependent) variables
L <- 4 #Number of levels in factor (treatment) 1
M <- 5 #Number of levels in factor (treatment) 2
N <- 100
X <- matrix(cbind(round(runif(N, 0.5, L+0.49)),round(runif(N,0.5,M+0.49))),
            N, J)
alpha <- runif(K,-1,1)
beta <- matrix(runif(K*L,-2,2), K, L)
beta[,L] <- -rowSums(beta[, -L])
gamma <- matrix(runif(K*M,-2,2), K, M)
gamma[,M] <- -rowSums(gamma[, -M])
Y <- matrix(NA,N,K)
for (k in 1:K) {
  Y[,k] <- alpha[k] + beta[k,X[,1]] + gamma[k,X[,2]] + rnorm(1,0,0.1)}
S <- diag(K)
mon.names <- c("LP", "s.o.beta", "s.o.gamma", "s.o.epsilon",
  parm.names(list(s.beta=rep(0,K), s.gamma=rep(0,K),
    s.epsilon=rep(0,K))))
parm.names <- parm.names(list(alpha=rep(0,K), beta=matrix(0,K,(L-1)),
  gamma=matrix(0,K,(M-1)), Omega=diag(K), log.sigma=rep(0,2)),
  uppertri=c(0,0,0,1,0))
MyData <- list(J=J, K=K, L=L, M=M, N=N, S=S, X=X, Y=Y,
  mon.names=mon.names, parm.names=parm.names)
```

### 43.3. Initial Values

```
Initial.Values <- c(rep(0,K), rep(0,K*(L-1)), rep(0,K*(M-1)),
  S[upper.tri(S, diag=TRUE)], rep(0,2))
```

### 43.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
```

```

alpha <- parm[grepl("alpha", Data$parm.names)]
beta <- matrix(c(parm[grepl("beta", Data$parm.names)], rep(0,K)),
  Data$K, Data$L)
beta[,L] <- -rowSums(beta[, -L])
gamma <- matrix(c(parm[grepl("gamma", Data$parm.names)], rep(0,K)),
  Data$K, Data$M)
gamma[,M] <- -rowSums(gamma[, -M])
Omega <- matrix(NA, Data$K, Data$K)
Omega[upper.tri(Omega, diag=TRUE)] <- parm[grepl("Omega",
  Data$parm.names)]
Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
Sigma <- solve(Omega)
sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
### Log(Prior Densities)
alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
beta.prior <- sum(dnorm(beta, 0, sigma[1], log=TRUE))
gamma.prior <- sum(dnorm(gamma, 0, sigma[2], log=TRUE))
Omega.prior <- dwishart(Omega, NROW(Data$S), Data$S, log=TRUE)
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
mu <- matrix(0, Data$N, Data$K)
for (k in 1:K) {
  mu[,k] <- alpha[k] + beta[k, Data$X[,1]] + gamma[k, Data$X[,2]]
}
LL <- sum(dmvn(Data$Y, mu, Sigma, log=TRUE))
### Variance Components, Omnibus
s.o.beta <- sd(as.vector(beta))
s.o.gamma <- sd(as.vector(gamma))
s.o.epsilon <- sd(as.vector(Data$Y - mu))
### Variance Components, Univariate
s.beta <- sd(t(beta))
s.gamma <- sd(t(gamma))
s.epsilon <- sd(Data$Y - mu)
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + gamma.prior + Omega.prior +
  sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, s.o.beta, s.o.gamma,
  s.o.epsilon, s.beta, s.gamma, s.epsilon), yhat=mu, parm=parm)
return(Modelout)
}

```

## 44. Mixture Model, Finite

This finite mixture model (FMM) imposes a multilevel structure on each of the  $J$  regression effects in  $\beta$ , so that mixture components share a common residual variance,  $\nu_j$ . Identifiability is gained at the expense of some shrinkage.

#### 44.1. Form

$$\begin{aligned}
\mathbf{y} &\sim \mathcal{N}(\mu_{1:N,m}, \sigma^2) \\
\mu_{1:N,m} &= \mathbf{X}\beta_{m,1:J}, \quad m = 1, \dots, M \\
\beta_{m,j} &\sim \mathcal{N}(0, \nu_j^2), \quad j = 1, \dots, J \\
\nu_j &\sim \mathcal{HC}(25) \\
\sigma &\sim \mathcal{HC}(25) \\
\pi_{1:M} &\sim \mathcal{D}(\alpha_{1:M}) \\
\pi_m &= \frac{\sum_{i=1}^N \delta_{i,m}}{\sum \delta} \\
\mathbf{p}_{i,m} &= \frac{\delta_{i,m}}{\sum_{m=1}^M \delta_{i,m}} \\
\delta_{i,m} &= \exp(\mathbf{X}\delta_{i,m}), \quad m = 1, \dots, (M-1) \\
\delta_{1:N,M} &= 1 \\
\delta_{i,m} &\sim \mathcal{N}(0, 1000) \in [-10, 10], \quad m = 1, \dots, (M-1) \\
\alpha_m &= 1
\end{aligned}$$

#### 44.2. Data

```

M <- 2 #Number of mixtures
alpha <- rep(1,M) #Prior probability of mixing probabilities
data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- c("LP", parm.names(list(pi=rep(0,M), sigma=0)))
parm.names <- parm.names(list(beta=matrix(0,M,J), log.nu=rep(0,J),
  log.delta=matrix(0,N,M-1), log.sigma=0))
MyData <- list(J=J, M=M, N=N, X=X, alpha=alpha, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

#### 44.3. Initial Values

```

Initial.Values <- c(runif(M*J), rep(0,J), runif(N*(M-1),-1,1), 0)

```

#### 44.4. Model

```

Model <- function(parm, Data)
{

```

```

### Parameters
beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$M, Data$J)
delta <- interval(parm[grepl("log.delta", Data$parm.names)], -10, 10)
parm[grepl("log.delta", Data$parm.names)] <- delta
delta <- matrix(c(exp(delta), rep(1, Data$N)), Data$N, Data$M)
pi <- colSums(delta) / sum(delta)
nu <- exp(parm[grepl("log.nu", Data$parm.names)])
sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, matrix(rep(nu, Data$M), Data$M,
  Data$J, byrow=TRUE), log=TRUE))
delta.prior <- sum(dtrunc(delta, "norm", a=exp(-10), b=exp(10),
  mean=log(1/Data$M), sd=sqrt(1000), log=TRUE))
pi.prior <- ddirichlet(pi, Data$alpha, log=TRUE)
nu.prior <- sum(dhalfcauchy(nu, 25, log=TRUE))
sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
p <- delta / rowSums(delta)
LL <- mu <- matrix(NA, Data$N, Data$M)
for (m in 1:M) {mu[,m] <- tcrossprod(beta[m,], Data$X)}
p <- apply(p, 1, which.max)
mu <- diag(mu[,p])
LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + delta.prior + pi.prior + nu.prior +
  sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,pi,sigma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 45. Mixture Model, Poisson-Gamma

### 45.1. Form

$$\mathbf{y} \sim \mathcal{P}(\lambda)$$

$$\lambda \sim \mathcal{G}(\alpha\mu, \alpha)$$

$$\mu = \exp(\mathbf{X}\beta)$$

$$\alpha \sim \mathcal{HC}(25)$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

## 45.2. Data

```

N <- 20
J <- 3
X <- matrix(runif(N*J,-2,2),N,J); X[,1] <- 1
beta <- runif(J,-2,2)
y <- as.vector(round(exp(tcrossprod(beta, X))))
mon.names <- "LP"
parm.names <- parm.names(list(log.alpha=0, beta=rep(0,J),
                             log.lambda=rep(0,N)))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)

```

## 45.3. Initial Values

```
Initial.Values <- c(0, rep(0,J), rep(0,N))
```

## 45.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  alpha <- exp(parm[grepl("log.alpha", Data$parm.names)])
  ### Parameters
  beta <- parm[grepl("beta", Data$parm.names)]
  lambda <- exp(parm[grepl("log.lambda", Data$parm.names)])
  mu <- exp(tcrossprod(beta, Data$X))
  ### Log(Hyperprior Densities)
  alpha.prior <- dhalfcauchy(alpha, 25, log=TRUE)
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  lambda.prior <- sum(dgamma(lambda, alpha*mu, alpha, log=TRUE))
  ### Log-Likelihood
  LL <- sum(dpois(Data$y, lambda, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior + lambda.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=lambda, parm=parm)
  return(Modelout)
}

```

# 46. Multinomial Logit

## 46.1. Form

$$\mathbf{y}_i \sim \text{CAT}(\mathbf{p}_{i,1:J}), \quad i = 1, \dots, N$$

$$\mathbf{p}_{i,j} = \frac{\phi_{i,j}}{\sum_{j=1}^J \phi_{i,j}}, \quad \sum_{j=1}^J \mathbf{p}_{i,j} = 1$$

$$\phi = \exp(\mu)$$

$$\mu_{i,J} = 0, \quad i = 1, \dots, N$$

$$\mu_{i,j} = \mathbf{X}_{i,1:K} \beta_{j,1:K} \in [-700, 700], \quad j = 1, \dots, (J-1)$$

$$\beta_{j,k} \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1), \quad k = 1, \dots, K$$

## 46.2. Data

```

y <- x01 <- x02 <- c(1:300)
y[1:100] <- 1
y[101:200] <- 2
y[201:300] <- 3
x01[1:100] <- rnorm(100, 25, 2.5)
x01[101:200] <- rnorm(100, 40, 4.0)
x01[201:300] <- rnorm(100, 35, 3.5)
x02[1:100] <- rnorm(100, 2.51, 0.25)
x02[101:200] <- rnorm(100, 2.01, 0.20)
x02[201:300] <- rnorm(100, 2.70, 0.27)
N <- length(y)
J <- 3 #Number of categories in y
K <- 3 #Number of predictors (including the intercept)
X <- matrix(c(rep(1,N),x01,x02),N,K)
mon.names <- "LP"
parm.names <- c("beta[1,1]", "beta[1,2]", "beta[1,3]", "beta[2,1]",
  "beta[2,2]", "beta[2,3]") ### Parameter Names [J,K]
MyData <- list(J=J, K=K, N=N, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

## 46.3. Initial Values

```
Initial.Values <- c(rep(0,(J-1)*K))
```

## 46.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:(Data$J-1*Data$K)]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- matrix(0,Data$N,Data$J)
  mu[,1] <- tcrossprod(beta[1:3], Data$X)

```

```

mu[,2] <- tcrossprod(beta[4:6], Data$X)
mu <- interval(mu, -700, 700)
phi <- exp(mu)
p <- phi / rowSums(phi)
LL <- sum(dcat(Data$y, p, log=TRUE))
yrep <- apply(p,1,which.max)
### Log-Posterior
LP <- LL + beta.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
return(Modelout)
}

```

## 47. Multinomial Logit, Nested

### 47.1. Form

$$\mathbf{y}_i \sim \mathcal{CAT}(\mathbf{P}_{i,1:J}), \quad i = 1, \dots, N$$

$$\mathbf{P}_{1:N,1} = \frac{\mathbf{R}}{\mathbf{R} + \exp(\alpha \mathbf{I})}$$

$$\mathbf{P}_{1:N,2} = \frac{(1 - \mathbf{P}_{1:N,1})\mathbf{S}_{1:N,1}}{\mathbf{V}}$$

$$\mathbf{P}_{1:N,3} = \frac{(1 - \mathbf{P}_{1:N,1})\mathbf{S}_{1:N,2}}{\mathbf{V}}$$

$$\mathbf{R}_{1:N} = \exp(\mu_{1:N,1})$$

$$\mathbf{S}_{1:N,1:2} = \exp(\mu_{1:N,2:3})$$

$$\mathbf{I} = \log(\mathbf{V})$$

$$\mathbf{V}_i = \sum_{k=1}^K \mathbf{S}_{i,k}, \quad i = 1, \dots, N$$

$$\mu_{1:N,1} = \mathbf{X}\boldsymbol{\iota} \in [-700, 700]$$

$$\mu_{1:N,2} = \mathbf{X}\boldsymbol{\beta}_{2,1:K} \in [-700, 700]$$

$$\boldsymbol{\iota} = \alpha \boldsymbol{\beta}_{1,1:K}$$

$$\alpha \sim \mathcal{EKP}(1) \in [0, 2]$$

$$\beta_{j,k} \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1) \quad k = 1, \dots, K$$

where there are  $J = 3$  categories of  $\mathbf{y}$ ,  $K = 3$  predictors,  $\mathbf{R}$  is the non-nested alternative,  $\mathbf{S}$  is the nested alternative,  $\mathbf{V}$  is the observed utility in the nest,  $\alpha$  is effectively 1 - correlation and has a truncated exponential distribution, and  $\boldsymbol{\iota}$  is a vector of regression effects for the isolated alternative after  $\alpha$  is taken into account. The third alternative is the reference category.

## 47.2. Data

```

y <- x01 <- x02 <- c(1:300)
y[1:100] <- 1
y[101:200] <- 2
y[201:300] <- 3
x01[1:100] <- rnorm(100, 25, 2.5)
x01[101:200] <- rnorm(100, 40, 4.0)
x01[201:300] <- rnorm(100, 35, 3.5)
x02[1:100] <- rnorm(100, 2.51, 0.25)
x02[101:200] <- rnorm(100, 2.01, 0.20)
x02[201:300] <- rnorm(100, 2.70, 0.27)
N <- length(y)
J <- 3 #Number of categories in y
K <- 3 #Number of predictors (including the intercept)
X <- matrix(c(rep(1,N),x01,x02),N,K)
mon.names <- c("LP",parm.names(list(iota=rep(0,K))))
parm.names <- parm.names(list(alpha=0, beta=matrix(0,J-1,K)))
MyData <- list(J=J, K=K, N=N, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

## 47.3. Initial Values

```
Initial.Values <- c(0.5, rep(0.1,(J-1)*K))
```

## 47.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  alpha.rate <- 1
  ### Parameters
  alpha <- interval(parm[1],0,2); parm[1] <- alpha
  beta <- matrix(parm[grep("beta", Data$parm.names)], Data$J-1, Data$K)
  ### Log(Prior Densities)
  alpha.prior <- dtrunc(alpha, "exp", a=0, b=2, rate=alpha.rate,
    log=TRUE)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- P <- matrix(0,Data$N,Data$J)
  iota <- alpha * beta[1,]
  mu[,1] <- tcrossprod(iota, Data$X)
  mu[,2] <- tcrossprod(beta[2,], Data$X)
  mu <- interval(mu, -700, 700)
  R <- exp(mu[,1])
  S <- exp(mu[,2:3])
  V <- rowSums(S)
  I <- log(V)
}

```

```

P[,1] <- R / (R + exp(alpha*I))
P[,2] <- (1 - P[,1]) * S[,1] / V
P[,3] <- (1 - P[,1]) * S[,2] / V
LL <- sum(dcat(Data$y, P, log=TRUE))
yrep <- apply(P,1,which.max)
### Log-Posterior
LP <- LL + alpha.prior + beta.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,iota), yhat=yrep,
  parm=parm)
return(Modelout)
}

```

## 48. Multinomial Probit

In this form of MNP, the  $\beta$  parameters are sum-to-zero constraints in the reference category, and covariance matrix  $\Sigma$  includes all  $J$  categories of  $\mathbf{y}$ .

Note that the parameters and initial values for the upper triangular elements of  $\Sigma$  are read in as  $\Sigma$ , though the diagonal is read in as  $\log(\Sigma)$ , but still denoted as  $\Sigma$ . Apologies for any confusion this causes, and the diagonal elements could each be renamed manually in `parm.names`. The only reason this difference exists is that I am unsure of how to program that in `parm.names` for all occasions.

### 48.1. Form

$$\begin{aligned}
\mathbf{Z}_{i,1:J} &\sim \mathcal{N}_J(\mu_{i,1:J}, \Sigma), \quad i = 1, \dots, N \\
\mathbf{Z}_{i,j} &\in \begin{cases} [0,10] & \text{if } \mathbf{y}_i = j \\ [-10,0] & \end{cases} \\
\mu_{1:N,j} &= \mathbf{X}\beta_{j,1:K} \\
\Sigma &\sim \mathcal{IW}(J, \mathbf{R}), \quad \mathbf{R} = \mathbf{I}_J, \quad \Sigma[1,1] = 1 \\
\beta_{j,k} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1), \quad k = 1, \dots, K \\
\beta_{J,k} &= -\sum_{j=1}^{J-1} \beta_{j,k} \\
\mathbf{Z}_{i,j} &\sim \mathcal{N}(0, 1000) \in [-10, 10]
\end{aligned}$$

### 48.2. Data

```

y <- x1 <- x2 <- c(1:30)
y[1:10] <- 1
y[11:20] <- 2
y[21:30] <- 3
x1[1:10] <- rnorm(10, 25, 2.5)

```

```

x1[11:20] <- rnorm(10, 40, 4.0)
x1[21:30] <- rnorm(10, 35, 3.5)
x2[1:10] <- rnorm(10, 2.51, 0.25)
x2[11:20] <- rnorm(10, 2.01, 0.20)
x2[21:30] <- rnorm(10, 2.70, 0.27)
N <- length(y)
J <- 3 #Number of categories in y
K <- 3 #Number of columns to be in design matrix X
R <- diag(J)
X <- matrix(c(rep(1,N),x1,x2),N,K)
mon.names <- "LP"
sigma.temp <- parm.names(list(Sigma=diag(J)), uppertri=1)
parm.names <- c(sigma.temp[2:length(sigma.temp)],
  parm.names(list(beta=matrix(0,(J-1),K), Z=matrix(0,N,J))))
MyData <- list(J=J, K=K, N=N, R=R, X=X, mon.names=mon.names,
  parm.names=parm.names, y=y)

```

### 48.3. Initial Values

```

Initial.Values <- c(rep(0,length(R[upper.tri(R, diag=TRUE)]))-1),
  rep(0,(J-1)*K), rep(0,N,J))

```

### 48.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$J-1, Data$K)
  beta <- rbind(beta, colSums(beta)*-1) #Sum to zero constraint
  Sigma <- matrix(NA, Data$J, Data$J)
  Sigma[upper.tri(Sigma, diag=TRUE)] <- c(0, parm[grepl("Sigma",
    Data$parm.names)])
  Sigma[lower.tri(Sigma)] <- Sigma[upper.tri(Sigma)]
  diag(Sigma) <- exp(diag(Sigma))
  Z <- matrix(parm[grepl("Z", Data$parm.names)], Data$N, Data$J)
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  Sigma.prior <- dinvwishart(Sigma, Data$J, Data$R, log=TRUE)
  Z.prior <- sum(dnorm(Z, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- matrix(0,Data$N,Data$J)
  for (j in 1:Data$J) {mu[,j] <- tcrossprod(beta[j,], Data$X)}
  Y <- indmat(Data$y)
  Z <- ifelse(Z > 10, 10, Z); Z <- ifelse({Y == 0} & {Z > 0}, 0, Z)
  Z <- ifelse(Z < -10, -10, Z); Z <- ifelse({Y == 1} & {Z < 0}, 0, Z)
}

```

```

parm[grep("Z", Data$parm.names)] <- as.vector(Z)
LL <- sum(dmvn(Z, mu, Sigma, log=TRUE))
yrep <- apply(Z, 1, which.max)
#eta <- exp(mu)
#p <- eta / rowSums(eta)
### Log-Posterior
LP <- LL + beta.prior + Sigma.prior + Z.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
return(Modelout)
}

```

## 49. Multivariate Binary Probit

### 49.1. Form

$$\begin{aligned}
\mathbf{Z}_{i,1:J} &\sim \mathcal{N}_J(\mu_{i,1:J}, \Sigma), \quad i = 1, \dots, N \\
\mathbf{Z}_{i,j} &\in \begin{cases} [0,10] & \text{if } \mathbf{y}_i = j \\ [-10,0] & \end{cases} \\
\mu_{1:N,j} &= \mathbf{X}\beta_{j,1:K} \\
\Sigma &\sim \mathcal{IW}(J, \mathbf{R}), \quad \mathbf{R} = \mathbf{I}_J, \quad \Sigma[1,1] = 1 \\
\beta_{j,k} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, (J-1), \quad k = 1, \dots, K \\
\beta_{J,k} &= -\sum_{j=1}^{J-1} \beta_{j,k} \\
\mathbf{Z}_{i,j} &\sim \mathcal{N}(0, 1000) \in [-10, 10]
\end{aligned}$$

### 49.2. Data

```

N <- 30
J <- 3 #Number of binary dependent variables
K <- 3 #Number of columns to be in design matrix X
Y <- matrix(round(runif(N*J)), N, J)
X <- matrix(1, N, K)
for (k in 2:K) {X[,k] <- rnorm(N, runif(1,-3,3), runif(1,0.1,3))}
R <- diag(J)
mon.names <- "LP"
sigma.temp <- parm.names(list(Sigma=diag(J)), uppertri=1)
parm.names <- c(sigma.temp[2:length(sigma.temp)],
  parm.names(list(beta=matrix(0, (J-1), K), Z=matrix(0, N, J))))
MyData <- list(J=J, K=K, N=N, R=R, X=X, Y=Y, mon.names=mon.names,

```

```
parm.names=parm.names)
```

### 49.3. Initial Values

```
Initial.Values <- c(rep(0,length(R[upper.tri(R, diag=TRUE)]))-1),      rep(0,(J-1)*K),
rep(0,N,J))
```

### 49.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$J-1, Data$K)
  beta <- rbind(beta, colSums(beta)*-1) #Sum to zero constraint
  Sigma <- matrix(NA, Data$J, Data$J)
  Sigma[upper.tri(Sigma, diag=TRUE)] <- c(0, parm[grepl("Sigma",
    Data$parm.names)])
  Sigma[lower.tri(Sigma)] <- Sigma[upper.tri(Sigma)]
  diag(Sigma) <- exp(diag(Sigma))
  Z <- matrix(parm[grepl("Z", Data$parm.names)], Data$N, Data$J)
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  Sigma.prior <- dinvwishart(Sigma, Data$J, Data$R, log=TRUE)
  Z.prior <- sum(dnorm(Z, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  mu <- matrix(0,Data$N,Data$J)
  for (j in 1:Data$J) mu[,j] <- tcrossprod(beta[j,], Data$X)
  Z <- ifelse(Z > 10, 10, Z)
  Z <- ifelse({Data$Y == 0} & {Z > 0}, 0, Z)
  Z <- ifelse(Z < -10, -10, Z)
  Z <- ifelse({Data$Y == 1} & {Z < 0}, 0, Z)
  parm[grepl("Z", Data$parm.names)] <- as.vector(Z)
  LL <- sum(dmvn(Z, mu, Sigma, log=TRUE))
  yrep <- ifelse(Z >= 0, 1, 0)
  ### Log-Posterior
  LP <- LL + beta.prior + Sigma.prior + Z.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=yrep, parm=parm)
  return(Modelout)
}
```

## 50. Multivariate Regression

### 50.1. Form

$$\begin{aligned} \mathbf{Y}_{i,k} &\sim \mathcal{N}_K(\mu_{i,k}, \Sigma), \quad i = 1, \dots, N; \quad k = 1, \dots, K \\ \mu_{i,k} &= \mathbf{X}_{1:N,k} \beta_{k,1:J} \\ \Sigma &= \Omega^{-1} \\ \Omega &\sim \mathcal{W}(K, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_K \\ \beta_{k,j} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \end{aligned}$$

### 50.2. Data

```

N <- 20
year <- c(1935,1936,1937,1938,1939,1940,1941,1942,1943,1944,1945,1946,
          1947,1948,1949,1950,1951,1952,1953,1954)
IG <- c(33.1,45.0,77.2,44.6,48.1,74.4,113.0,91.9,61.3,56.8,93.6,159.9,
        147.2,146.3,98.3,93.5,135.2,157.3,179.5,189.6)
VG <- c(1170.6,2015.8,2803.3,2039.7,2256.2,2132.2,1834.1,1588.0,1749.4,
        1687.2,2007.7,2208.3,1656.7,1604.4,1431.8,1610.5,1819.4,2079.7,
        2371.6,2759.9)
CG <- c(97.8,104.4,118.0,156.2,172.6,186.6,220.9,287.8,319.9,321.3,319.6,
        346.0,456.4,543.4,618.3,647.4,671.3,726.1,800.3,888.9)
IW <- c(12.93,25.90,35.05,22.89,18.84,28.57,48.51,43.34,37.02,37.81,
        39.27,53.46,55.56,49.56,32.04,32.24,54.38,71.78,90.08,68.60)
VW <- c(191.5,516.0,729.0,560.4,519.9,628.5,537.1,561.2,617.2,626.7,
        737.2,760.5,581.4,662.3,583.8,635.2,723.8,864.1,1193.5,1188.9)
CW <- c(1.8,0.8,7.4,18.1,23.5,26.5,36.2,60.8,84.4,91.2,92.4,86.0,111.1,
        130.6,141.8,136.7,129.7,145.5,174.8,213.5)
J <- 4 #Number of columns in design matrix
K <- 2 #Number of DVs
Y <- matrix(c(IG,IW),N,K)
X <- matrix(c(VG,CG,VW,CW),N,J)
S <- diag(ncol(Y))
mon.names <- "LP"
parm.names <- parm.names(list(beta=matrix(0,K,J), Omega=diag(K)),
                          uppertri=c(0,1))
MyData <- list(J=J, K=K, N=N, S=S, X=X, Y=Y, mon.names=mon.names,
               parm.names=parm.names)

```

### 50.3. Initial Values

### 50.4. Data

```

Model <- function(parm, Data)
{

```

```

#### Parameters
beta <- matrix(parm[grepl("beta", Data$parm.names)], Data$K, Data$J)
Omega <- matrix(NA, Data$K, Data$K)
Omega[upper.tri(Omega, diag=TRUE)] <- parm[grepl("Omega",
  Data$parm.names)]
Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
Sigma <- solve(Omega)
#### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
Omega.prior <- dwishart(Omega, Data$K, Data$S, log=TRUE)
#### Log-Likelihood
mu <- matrix(0, Data$N, Data$K)
for (k in 1:Data$K) {mu[,k] <- tcrossprod(beta[k,], Data$X)}
LL <- sum(dmvn(Data$Y, mu, Sigma, log=TRUE))
#### Log-Posterior
LP <- LL + beta.prior + Omega.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
return(Modelout)
}

```

## 51. Normal, Multilevel

This is Gelman's school example (Gelman, Carlin, Stern, and Rubin 2004). Note that **LaplacesDemon** is much slower to converge compared to this example that uses the **R2WinBUGS** package (Gelman 2011), an R package on CRAN. However, also note that Laplace's Demon (eventually) provides a better answer (higher ESS, lower DIC, etc.).

### 51.1. Form

$$\begin{aligned}
 \mathbf{y}_j &\sim \mathcal{N}(\theta_j, \tau_j^{-1}), \quad j = 1, \dots, J \\
 \theta_j &\sim \mathcal{N}(\theta_\mu, \theta_\tau^{-1}), \quad j = 1, \dots, J \\
 \theta_\mu &\sim \mathcal{N}(0, 1000) \\
 \theta_\tau &= \frac{1}{\theta_\sigma^2} \\
 \sigma &\sim \mathcal{U}(1.0E - 100, 100) \\
 \tau_j &= \sigma_j^{-2}, \quad j = 1, \dots, J
 \end{aligned}$$

### 51.2. Data

```

J <- 8
y <- c(28.4, 7.9, -2.8, 6.8, -0.6, 0.6, 18.0, 12.2)
sd <- c(14.9, 10.2, 16.3, 11.0, 9.4, 11.4, 10.4, 17.6)

```

```

mon.names <- c("LP","theta.tau")
parm.names <- parm.names(list(theta=rep(0,J), theta.mu=0, sigma=0))
MyData <- list(J=J, mon.names=mon.names, parm.names=parm.names, sd=sd, y=y)

```

### 51.3. Initial Values

```
Initial.Values <- c(rep(0,J), 0, 1)
```

### 51.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  theta.mu <- parm[Data$J+1]
  sigma <- interval(parm[grep("sigma", Data$parm.names)], 1.0E-100, 100)
  parm[grep("sigma", Data$parm.names)] <- sigma
  theta.tau <- 1 / sigma^2
  tau.alpha <- 1.0E-3
  tau.beta <- 1.0E-3
  ### Parameters
  theta <- parm[1:Data$J]; tau <- 1/(Data$sd*Data$sd)
  ### Log(Hyperprior and Prior Densities)
  theta.mu.prior <- dnorm(theta.mu, sqrt(1000), log=TRUE)
  sigma.prior <- dunif(sigma, 1.0E-100, 100, log=TRUE)
  tau.prior <- sum(dgamma(tau, tau.alpha, tau.beta, log=TRUE))
  theta.prior <- sum(dnorm(theta, theta.mu, 1/sqrt(theta.tau), log=TRUE))
  ### Log-Likelihood
  LL <- sum(dnorm(Data$y, theta, 1/sqrt(tau), log=TRUE))
  ### Log-Posterior
  LP <- LL + theta.mu.prior + sigma.prior + theta.prior + tau.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, theta.tau),
    yhat=theta, parm=parm)
  return(Modelout)
}

```

## 52. Panel, Autoregressive Poisson

### 52.1. Form

$$\mathbf{Y} \sim \mathcal{P}(\Lambda)$$

$$\Lambda_{1:N,1} = \exp(\alpha + \beta \mathbf{x})$$

$$\Lambda_{1:N,t} = \exp(\alpha + \beta \mathbf{x} + \rho \log(\mathbf{Y}_{1:N,t-1})), \quad t = 2, \dots, T$$

$$\begin{aligned}\alpha_i &\sim \mathcal{N}(\alpha_\mu, \alpha_\sigma^2), \quad i = 1, \dots, N \\ \alpha_\mu &\sim \mathcal{N}(0, 1000) \\ \alpha_\sigma &\sim \mathcal{HC}(25) \\ \beta &\sim \mathcal{N}(0, 1000) \\ \rho &\sim \mathcal{N}(0, 1000)\end{aligned}$$

## 52.2. Data

```

N <- 10
T <- 10
alpha <- rnorm(N, 2, 0.5)
rho <- 0.5
beta <- 0.5
x <- runif(N, 0, 1)
Y <- matrix(NA, N, T)
Y[,1] <- exp(alpha + beta*x)
for (t in 2:T) {Y[,t] <- exp(alpha + beta*x + rho*log(Y[,t-1]))}
Y <- round(Y)
mon.names <- c("LP", "alpha.sigma")
parm.names <- parm.names(list(alpha=rep(0, N), alpha.mu=0,
  log.alpha.sigma=0, beta=0, rho=0))
MyData <- list(N=N, T=T, Y=Y, mon.names=mon.names, parm.names=parm.names,
  x=x)

```

## 52.3. Initial Values

```
Initial.Values <- c(rep(0, N), 0, log(1), 0, 0)
```

## 52.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  alpha.mu <- parm[Data$N+1]
  alpha.sigma <- exp(parm[Data$N+2])
  ### Parameters
  alpha <- parm[1:Data$N]
  beta <- parm[grep("beta", Data$parm.names)]
  rho <- parm[grep("rho", Data$parm.names)]
  ### Log(Hyperprior and Prior Densities)
  alpha.mu.prior <- dnorm(alpha.mu, 0, sqrt(1000), log=TRUE)
  alpha.sigma.prior <- dhalfcauchy(alpha.sigma, 25, log=TRUE)
  alpha.prior <- sum(dnorm(alpha, alpha.mu, alpha.sigma, log=TRUE))
  beta.prior <- dnorm(beta, 0, sqrt(1000), log=TRUE)
}

```

```

rho.prior <- dnorm(rho, 0, sqrt(1000), log=TRUE)
### Log-Likelihood
Lambda <- Data$Y
Lambda[,1] <- exp(alpha + beta*x)
Lambda[,2:Data$T] <- exp(alpha + beta*Data$x +
  rho*log(Data$Y[,1:(Data$T-1)]))
LL <- sum(dpois(Data$Y, Lambda, log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + alpha.mu.prior + alpha.sigma.prior +
  beta.prior + rho.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,alpha.sigma),
  yhat=Lambda, parm=parm)
return(Modelout)
}

```

## 53. Penalized Spline Regression

This example is adapted from [Crainiceanu, Ruppert, and Wand \(2005\)](#). The user specifies the degree  $D$  of polynomials and the number  $K$  of knots. Regression effects  $\beta$  regard the polynomial in design matrix  $\mathbf{X}$ , and  $\gamma$  regard the splines in design matrix  $\mathbf{S}$ .

$$\begin{aligned}
 \mathbf{y} &\sim \mathcal{N}(\mu, \sigma_1^2) \\
 \mu &= \mathbf{X}\beta + \mathbf{S}\gamma \\
 \mathbf{S}_{i,k} &= \begin{cases} (\mathbf{x}_i - k)^D & \text{if } \mathbf{S}_{i,k} > 0 \\ 0 & \text{otherwise} \end{cases} \\
 \mathbf{X}_{i,d} &= \mathbf{x}_i^{d-1}, \quad d = 2, \dots, (D+1) \\
 \mathbf{X}_{i,1} &= 1 \\
 \beta_d &\sim \mathcal{N}(0, 1000), \quad d = 1, \dots, (D+1) \\
 \gamma_k &\sim \mathcal{N}(0, \sigma_2^2), \quad k = 1, \dots, K \\
 \sigma_j &\sim \mathcal{HC}(25), \quad j = 1, \dots, 2
 \end{aligned}$$

### 53.1. Form

### 53.2. Data

```

data(demonsnacks)
N <- NROW(demonsnacks)
K <- 10 #Number of knots
D <- 2 #Degree of polynomial
y <- log(demonsnacks$Calories)
x <- demonsnacks[,7]

```

```

k <- as.vector(quantile(x, probs=(1:K / (K+1))))
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,D+1), gamma=rep(0,K),
  log.sigma=rep(0,2)))
MyData <- list(D=D, K=K, N=N, mon.names=mon.names,
  parm.names=parm.names, k=k, x=x, y=y)

```

### 53.3. Initial Values

```
Initial.Values <- c(rep(0,D+1), rep(0,K), log(c(1,1)))
```

### 53.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[grepl("beta", Data$parm.names)]
  gamma <- parm[grepl("gamma", Data$parm.names)]
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  gamma.prior <- sum(dnorm(gamma, 0, sigma[2], log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  X <- matrix(Data$x, Data$N, Data$D)
  for (d in 2:Data$D) X[,d] <- X[,d]^d
  X <- cbind(1,X)
  S <- matrix(Data$x, Data$N, Data$K) -
    matrix(Data$k, Data$N, Data$K, byrow=TRUE)
  S <- ifelse(S > 0, S, 0)
  S <- S^Data$D
  mu <- tcrossprod(beta, X) + tcrossprod(gamma, S)
  LL <- sum(dnorm(Data$y, mu, sigma[1], log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + gamma.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
  return(Modelout)
}

```

## 54. Poisson Regression

### 54.1. Form

$$\begin{aligned} \mathbf{y} &\sim \mathcal{P}(\lambda) \\ \lambda &= \exp(\mathbf{X}\boldsymbol{\beta}) \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \end{aligned}$$

### 54.2. Data

```
N <- 10000
J <- 5
X <- matrix(runif(N*J,-2,2),N,J); X[,1] <- 1
beta <- runif(J,-2,2)
y <- as.vector(round(exp(tcrossprod(beta, X))))
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,J)))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

### 54.3. Initial Values

```
Initial.Values <- rep(0,J)
```

### 54.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  lambda <- exp(tcrossprod(beta, Data$X))
  LL <- sum(dpois(Data$y, lambda, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP,
    yhat=lambda, parm=parm)
  return(Modelout)
}
```

## 55. Polynomial Regression

In this univariate example, the degree of the polynomial is specified as  $D$ . For a more robust extension to estimating nonlinear relationships between  $\mathbf{y}$  and  $\mathbf{x}$ , see penalized spline regression in section [53](#).

### 55.1. Form

$$\begin{aligned} \mathbf{y} &\sim \mathcal{N}(\mu, \sigma^2) \\ \mu &= \mathbf{X}\beta \\ \mathbf{X}_{i,d} &= \mathbf{x}_i^{d-1}, \quad d = 1, \dots, (D+1) \\ \mathbf{X}_{i,1} &= 1 \\ \beta_d &\sim \mathcal{N}(0, 1000), \quad d = 1, \dots, (D+1) \\ \sigma &\sim \mathcal{HC}(25) \end{aligned}$$

### 55.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
D <- 2 #Degree of polynomial
y <- log(demonsnacks$Calories)
x <- demonsnacks[,7]
mon.names <- "LP"
parm.names <- parm.names(list(beta=rep(0,D+1), log.sigma=0))
MyData <- list(D=D, N=N, mon.names=mon.names, parm.names=parm.names, x=x,
               y=y)
```

### 55.3. Initial Values

```
Initial.Values <- c(rep(0,D+1), log(1))
```

### 55.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[grepl("beta", Data$parm.names)]
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  X <- matrix(Data$x, Data$N, Data$D)
  for (d in 2:Data$D) {X[,d] <- X[,d]^d}
  X <- cbind(1,X)
  mu <- tcrossprod(beta, X)
  LL <- sum(dnorm(Data$y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + sigma.prior
```

```
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
return(Modelout)
}
```

## 56. Proportional Hazards Regression, Weibull

Although the dependent variable is usually denoted as **t** in survival analysis, it is denoted here as **y** so Laplace's Demon recognizes it as a dependent variable for posterior predictive checks. This example does not support censoring, but it will be included soon.

### 56.1. Form

$$\begin{aligned} \mathbf{y}_i &\sim \mathcal{WETB}(\gamma, \mu_i), \quad i = 1, \dots, N \\ \mu &= \exp(\mathbf{X}\beta) \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \gamma &\sim \mathcal{G}(1, 0.001) \end{aligned}$$

### 56.2. Data

```
N <- 50
J <- 5
X <- matrix(runif(N*J,-2,2),N,J); X[,1] <- 1
beta <- c(1,runif(J-1,-1,1))
y <- as.vector(round(exp(tcrossprod(beta, X)))) + 1 # Undefined at zero
mon.names <- c("LP","gamma")
parm.names <- parm.names(list(beta=rep(0,J), log.gamma=0))
MyData <- list(J=J, N=N, X=X, mon.names=mon.names, parm.names=parm.names,
  y=y)
```

### 56.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 56.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  gamma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
```

```

gamma.prior <- dgamma(gamma, 1, 1.0E-3, log=TRUE)
### Log-Likelihood
mu <- exp(tcrossprod(beta, Data$X))
LL <- sum(dweibull(Data$y, gamma, mu, log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + gamma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, gamma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 57. Revision, Normal

This example provides both an analytic solution and numerical approximation of the revision of a normal distribution. Given a normal prior distribution ( $\alpha$ ) and data distribution ( $\beta$ ), the posterior ( $\gamma$ ) is the revised normal distribution. This is an introductory example of Bayesian inference, and allows the user to experiment numerical approximation, such as with MCMC in *LaplaceDemon*. Note that, regardless of the data sample size  $N$  in this example, Laplace Approximation is inappropriate due to asymptotics since the data ( $\beta$ ) is perceived by the algorithm as a single datum rather than a collection of data. MCMC, on the other hand, is biased only by the effective number of samples taken of the posterior.

### Analytic Solution

```

prior.mu <- 0
prior.sigma <- 10
N <- 10
data.mu <- 1
data.sigma <- 2
posterior.mu <- (prior.sigma^-2 * prior.mu + N * data.sigma^-2 * data.mu) /
  (prior.sigma^-2 + N * data.sigma^-2)
posterior.sigma <- sqrt(1/(prior.sigma^-2 + data.sigma^-2))
posterior.mu
posterior.sigma

```

### 57.1. Form

$$\begin{aligned}
 \alpha &\sim \mathcal{N}(0, 10) \\
 \beta &\sim \mathcal{N}(1, 2) \\
 \gamma &= \frac{\alpha_{\sigma}^{-2}\alpha + N\beta_{\sigma}^{-2}\beta}{\alpha_{\sigma}^{-2} + N\beta_{\sigma}^{-2}}
 \end{aligned}$$

### 57.2. Data

```
N <- 10
```

```
mon.names <- c("LP","gamma")
parm.names <- c("alpha","beta")
MyData <- list(N=N, mon.names=mon.names, parm.names=parm.names)
```

### 57.3. Initial Values

```
Initial.Values <- c(0,0)
```

### 57.4. Model

```
Model <- function(parm, Data)
{
  ### Hyperparameters
  alpha.mu <- 0
  alpha.sigma <- 10
  beta.mu <- 1
  beta.sigma <- 2
  ### Parameters
  alpha <- parm[1]
  beta <- parm[2]
  ### Log(Prior Density)
  alpha.prior <- dnorm(alpha, alpha.mu, alpha.sigma, log=TRUE)
  ### Log-Likelihood Density
  LL <- dnorm(beta, beta.mu, beta.sigma, log=TRUE)
  ### Posterior
  gamma <- (alpha.sigma^-2 * alpha + N * beta.sigma^-2 * beta) /
    (alpha.sigma^-2 + N * beta.sigma^-2)
  ### Log(Posterior Density)
  LP <- LL + alpha.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,gamma), yhat=LL,
    parm=parm)
  return(Modelout)
}
```

## 58. Robust Regression

By replacing the normal distribution with the Student  $t$  distribution, linear regression is often called robust regression. As an alternative approach to robust regression, consider Laplace regression (see section 36).

### 58.1. Form

$$\mathbf{y} \sim t(\mu, \sigma^2, \nu)$$

$$\mu = \mathbf{X}\beta$$

$$\begin{aligned}\beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \sigma &\sim \mathcal{HC}(25) \\ \nu &\sim \mathcal{HC}(25)\end{aligned}$$

## 58.2. Data

```
N <- 10000
J <- 5
X <- matrix(1,N,J)
for (j in 2:J) {X[,j] <- rnorm(N,runif(1,-3,3),runif(1,0.1,1))}
beta <- runif(J,-3,3)
e <- rnorm(N,0,0.1)
y <- as.vector(tcrossprod(beta, X) + e)
mon.names <- c("LP", "sigma", "nu")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0, log.nu=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

## 58.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1), log(2))
```

## 58.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  nu <- exp(parm[Data$J+2])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  nu.prior <- dhalfcauchy(nu, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(dst(Data$y, mu, sigma, nu, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + sigma.prior + nu.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,nu), yhat=mu,
    parm=parm)
  return(Modelout)
}
```

## 59. Seemingly Unrelated Regression (SUR)

The following data was used by Zellner (1962) when introducing the Seemingly Unrelated Regression methodology.

### 59.1. Form

$$\begin{aligned}\mathbf{Y}_{t,k} &\sim \mathcal{N}_K(\mu_{t,k}, \Sigma), \quad t = 1, \dots, T; \quad k = 1, \dots, K \\ \mu_{1,t} &= \alpha_1 + \alpha_2 \mathbf{X}_{t,1} + \alpha_3 \mathbf{X}_{t,2}, \quad t = 1, \dots, T \\ \mu_{2,t} &= \beta_1 + \beta_2 \mathbf{X}_{t,3} + \beta_3 \mathbf{X}_{t,4}, \quad t = 1, \dots, T \\ \Sigma &= \Omega^{-1} \\ \Omega &\sim \mathcal{W}(K, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_K \\ \alpha_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J \\ \beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J\end{aligned}$$

where J=3, K=2, and T=20.

### 59.2. Data

```
T <- 20
year <- c(1935, 1936, 1937, 1938, 1939, 1940, 1941, 1942, 1943, 1944, 1945, 1946,
          1947, 1948, 1949, 1950, 1951, 1952, 1953, 1954)
IG <- c(33.1, 45.0, 77.2, 44.6, 48.1, 74.4, 113.0, 91.9, 61.3, 56.8, 93.6, 159.9,
        147.2, 146.3, 98.3, 93.5, 135.2, 157.3, 179.5, 189.6)
VG <- c(1170.6, 2015.8, 2803.3, 2039.7, 2256.2, 2132.2, 1834.1, 1588.0, 1749.4,
        1687.2, 2007.7, 2208.3, 1656.7, 1604.4, 1431.8, 1610.5, 1819.4, 2079.7,
        2371.6, 2759.9)
CG <- c(97.8, 104.4, 118.0, 156.2, 172.6, 186.6, 220.9, 287.8, 319.9, 321.3, 319.6,
        346.0, 456.4, 543.4, 618.3, 647.4, 671.3, 726.1, 800.3, 888.9)
IW <- c(12.93, 25.90, 35.05, 22.89, 18.84, 28.57, 48.51, 43.34, 37.02, 37.81,
        39.27, 53.46, 55.56, 49.56, 32.04, 32.24, 54.38, 71.78, 90.08, 68.60)
VW <- c(191.5, 516.0, 729.0, 560.4, 519.9, 628.5, 537.1, 561.2, 617.2, 626.7,
        737.2, 760.5, 581.4, 662.3, 583.8, 635.2, 723.8, 864.1, 1193.5, 1188.9)
CW <- c(1.8, 0.8, 7.4, 18.1, 23.5, 26.5, 36.2, 60.8, 84.4, 91.2, 92.4, 86.0, 111.1,
        130.6, 141.8, 136.7, 129.7, 145.5, 174.8, 213.5)
Y <- matrix(c(IG, IW), T, 2)
S <- diag(NCOL(Y))
mon.names <- c("LP", "Sigma[1,1]", "Sigma[2,1]", "Sigma[1,2]", "Sigma[2,2]")
parm.names <- parm.names(list(alpha=rep(0,3), beta=rep(0,3),
                             Omega=diag(2)), uppertri=c(0,0,1))
MyData <- list(S=S, T=T, Y=Y, CG=CG, CW=CW, IG=IG, IW=IW, VG=VG, VW=VW,
               mon.names=mon.names, parm.names=parm.names)
```

### 59.3. Initial Values

```
Initial.Values <- c(rep(0,3), rep(0,3), S[upper.tri(S, diag=TRUE)])
```

### 59.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1:3]
  beta <- parm[4:6]
  Omega <- matrix(parm[c(7,8,8,9)], NROW(Data$S), NROW(Data$S))
  Sigma <- solve(Omega)
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  Omega.prior <- dwishart(Omega, NROW(Data$S), Data$S, log=TRUE)
  ### Log-Likelihood
  mu <- matrix(0, Data$T, 2)
  mu[,1] <- alpha[1] + alpha[2]*Data$CG + alpha[3]*Data$VG
  mu[,2] <- beta[1] + beta[2]*Data$CW + beta[3]*Data$VW
  LL <- sum(dmvn(Data$Y, mu, Sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + alpha.prior + beta.prior + Omega.prior
  Modelout <- list(LP=LP, Dev=-2*LL,
    Monitor=c(LP, as.vector(Sigma)), yhat=mu, parm=parm)
  return(Modelout)
}
```

## 60. Simultaneous Equations

This example of simultaneous equations uses Klein's Model I ([Kleine 1950](#)) regarding economic fluctuations in the United States in 1920-1941 ( $N=22$ ). Usually, this example is modeled with 3-stage least squares (3SLS), excluding the uncertainty from multiple stages. By constraining each element in the instrumental variables matrix  $\nu \in [-10, 10]$ , this example estimates the model without resorting to stages. The dependent variable is matrix  $\mathbf{Y}$ , in which  $\mathbf{Y}_{1,1:N}$  is **C** or Consumption,  $\mathbf{Y}_{2,1:N}$  is **I** or Investment, and  $\mathbf{Y}_{3,1:N}$  is **Wp** or Private Wages. Here is a data dictionary:

```
A = Time Trend measured as years from 1931
C = Consumption
G = Government Nonwage Spending
I = Investment
K = Capital Stock
P = Private (Corporate) Profits
T = Indirect Business Taxes Plus Neg Exports
Wg = Government Wage Bill
Wp = Private Wages
X = Equilibrium Demand (GNP)
```

See [Kleine \(1950\)](#) for more information.

## 60.1. Form

$$\begin{aligned}
\mathbf{Y} &\sim \mathcal{N}(\mu, \Sigma) \\
\mu_{1,1} &= \alpha_1 + \alpha_2 \nu_{1,1} + \alpha_4 \nu_{2,1} \\
\mu_{1,i} &= \alpha_1 + \alpha_2 \nu_{1,i} + \alpha_3 \mathbf{P}_{i-1} + \alpha_4 \nu_{2,i}, \quad i = 2, \dots, N \\
\mu_{2,1} &= \beta_1 + \beta_2 \nu_{1,1} + \beta_4 \mathbf{K}_1 \\
\mu_{2,i} &= \beta_1 + \beta_2 \nu_{1,i} + \beta_3 \mathbf{P}_{i-1} + \beta_4 \mathbf{K}_i, \quad i = 2, \dots, N \\
\mu_{3,1} &= \gamma_1 + \gamma_2 \nu_{3,1} + \gamma_4 \mathbf{A}_1 \\
\mu_{3,i} &= \gamma_1 + \gamma_2 \nu_{3,i} + \gamma_3 \mathbf{X}_{i-1} + \gamma_4 \mathbf{A}_i, \quad i = 2, \dots, N \\
\mathbf{Z}_{j,i} &\sim \mathcal{N}(\nu_{j,i}, \sigma_j^2), \quad j = 1, \dots, 3 \\
\nu_{j,1} &= \pi_{j,1} + \pi_{j,3} \mathbf{K}_1 + \pi_{j,5} \mathbf{A}_1 + \pi_{j,6} \mathbf{T}_1 + \pi_{j,7} \mathbf{G}_1, \quad j = 1, \dots, 3 \\
\nu_{j,i} &= \pi_{j,1} + \pi_{j,2} \mathbf{P}_{i-1} + \pi_{j,3} \mathbf{K}_i + \pi_{j,4} \mathbf{X}_{i-1} + \pi_{j,5} \mathbf{A}_i + \pi_{j,6} \mathbf{T}_i + \pi_{j,7} \mathbf{G}_i, \quad i = 1, \dots, N, \quad j = 1, \dots, 3 \\
\alpha_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 4 \\
\beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 4 \\
\gamma_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 4 \\
\pi_{j,i} &\sim \mathcal{N}(0, 1000) \in [-10, 10], \quad j = 1, \dots, 3, \quad i = 1, \dots, N \\
\sigma_j &\sim \mathcal{HC}(25), \quad j = 1, \dots, 3 \\
\Omega &\sim \mathcal{W}(N, \mathbf{S}), \quad \mathbf{S} = \mathbf{I}_3 \\
\Sigma &= \Omega^{-1}
\end{aligned}$$

## 60.2. Data

```

N <- 22
A <- c(-11,-10,-9,-8,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10)
C <- c(39.8,41.9,45,49.2,50.6,52.6,55.1,56.2,57.3,57.8,55,50.9,45.6,46.5,
      48.7,51.3,57.7,58.7,57.5,61.6,65,69.7)
G <- c(2.4,3.9,3.2,2.8,3.5,3.3,3.3,4,4.2,4.1,5.2,5.9,4.9,3.7,4,4.4,2.9,4.3,
      5.3,6.6,7.4,13.8)
I <- c(2.7,-0.2,1.9,5.2,3,5.1,5.6,4.2,3,5.1,1,-3.4,-6.2,-5.1,-3,-1.3,2.1,2,
      -1.9,1.3,3.3,4.9)
K <- c(180.1,182.8,182.6,184.5,189.7,192.7,197.8,203.4,207.6,210.6,215.7,
      216.7,213.3,207.1,202,199,197.7,199.8,201.8,199.9,201.2,204.5)
P <- c(12.7,12.4,16.9,18.4,19.4,20.1,19.6,19.8,21.1,21.7,15.6,11.4,7,11.2,
      12.3,14,17.6,17.3,15.3,19,21.1,23.5)
T <- c(3.4,7.7,3.9,4.7,3.8,5.5,7,6.7,4.2,4,7.7,7.5,8.3,5.4,6.8,7.2,8.3,6.7,
      7.4,8.9,9.6,11.6)
Wg <- c(2.2,2.7,2.9,2.9,3.1,3.2,3.3,3.6,3.7,4,4.2,4.8,5.3,5.6,6,6.1,7.4,
      6.7,7.7,7.8,8,8.5)
Wp <- c(28.8,25.5,29.3,34.1,33.9,35.4,37.4,37.9,39.2,41.3,37.9,34.5,29,28.5,

```

```

      30.6,33.2,36.8,41,38.2,41.6,45,53.3)
X <- c(44.9,45.6,50.1,57.2,57.1,61,64,64.4,64.5,67,61.2,53.4,44.3,45.1,
      49.7,54.4,62.7,65,60.9,69.5,75.7,88.4)
year <- c(1920,1921,1922,1923,1924,1925,1926,1927,1928,1929,1930,1931,1932,
      1933,1934,1935,1936,1937,1938,1939,1940,1941)
Y <- matrix(c(C,I,Wp),3,N, byrow=TRUE)
Z <- matrix(c(P, Wp+Wg, X), 3, N, byrow=TRUE)
S <- diag(NROW(Y))
mon.names <- "LP"
parm.names <- parm.names(list(alpha=rep(0,4), beta=rep(0,4),
      gamma=rep(0,4), pi=matrix(0,3,7), log.sigma=rep(0,3),
      Omega=diag(3)), uppertri=c(0,0,0,0,0,1))
MyData <- list(A=A, C=C, G=G, I=I, K=K, N=N, P=P, S=S, T=T, Wg=Wg, Wp=Wp,
      X=X, Y=Y, Z=Z, mon.names=mon.names, parm.names=parm.names)

```

### 60.3. Initial Values

```

Initial.Values <- c(rep(0,4), rep(0,4), rep(0,4), rep(0,3*7), rep(0,3),      S[upper.tri(S,
diag=TRUE)])

```

### 60.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1:4]; beta <- parm[5:8]; gamma <- parm[9:12]
  pi <- matrix(interval(parm[grep("pi", Data$parm.names)],-10,10), 3, 7)
  parm[grep("pi", Data$parm.names)] <- as.vector(pi)
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  Omega <- matrix(NA, 3, 3)
  Omega[upper.tri(Omega, diag=TRUE)] <- parm[grep("Omega",
    Data$parm.names)]
  Omega[lower.tri(Omega)] <- Omega[upper.tri(Omega)]
  Sigma <- solve(Omega)
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  gamma.prior <- sum(dnorm(gamma, 0, sqrt(1000), log=TRUE))
  pi.prior <- sum(dnorm(pi, 0, sqrt(1000), log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  Omega.prior <- dwishart(Omega, NROW(Data$S), Data$S, log=TRUE)
  ### Log-Likelihood
  mu <- nu <- matrix(0,3,Data$N)
  for (i in 1:3) {
    nu[i,1] <- pi[i,1] + pi[i,3]*Data$K[1] + pi[i,5]*Data$A[1] +
      pi[i,6]*Data$T[1] + pi[i,7]*Data$G[1]

```

```

nu[i,-1] <- pi[i,1] + pi[i,2]*Data$P[-Data$N] +
  pi[i,3]*Data$K[-1] + pi[i,4]*Data$X[-Data$N] +
  pi[i,5]*Data$A[-1] + pi[i,6]*Data$T[-1] +
  pi[i,7]*Data$G[-1]}
LL <- sum(dnorm(Data$Z, nu, matrix(sigma, 3, Data$N), log=TRUE))
mu[1,1] <- alpha[1] + alpha[2]*nu[1,1] + alpha[4]*nu[2,1]
mu[1,-1] <- alpha[1] + alpha[2]*nu[1,-1] +
  alpha[3]*Data$P[-Data$N] + alpha[4]*nu[2,-1]
mu[2,1] <- beta[1] + beta[2]*nu[1,1] + beta[4]*Data$K[1]
mu[2,-1] <- beta[1] + beta[2]*nu[1,-1] +
  beta[3]*Data$P[-Data$N] + beta[4]*Data$K[-1]
mu[3,1] <- gamma[1] + gamma[2]*nu[3,1] + gamma[4]*Data$A[1]
mu[3,-1] <- gamma[1] + gamma[2]*nu[3,-1] +
  gamma[3]*Data$X[-Data$N] + gamma[4]*Data$A[-1]
LL2 <- sum(dmvn(t(Data$Y), t(mu), Sigma, log=TRUE))
if(!is.nan(LL2)) LL <- LL + LL2
### Log-Posterior
LP <- LL + alpha.prior + beta.prior + gamma.prior + pi.prior +
  sigma.prior + Omega.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=LP, yhat=mu, parm=parm)
return(Modelout)
}

```

## 61. Space-Time, Dynamic

This approach to space-time or spatiotemporal modeling applies kriging to a stationary spatial component for points in space  $s = 1, \dots, S$  first at time  $t = 1$ , where space is continuous and time is discrete. Vector  $\zeta$  contains these spatial effects. Next, SSM (State-Space Model) or DLM (Dynamic Linear Model) components are applied to the spatial parameters ( $\phi$ ,  $\kappa$ , and  $\lambda$ ) and regression effects ( $\beta$ ). These parameters are allowed to vary dynamically with time  $t = 2, \dots, T$ , and the resulting spatial process is estimated for each of these time-periods. When time is discrete, a dyanmic space-time process can be applied. The matrix  $\Theta$  contains the dynamically varying stationary spatial effects, or space-time effects. Spatial coordinates are given in longitude and latitude for  $s = 1, \dots, S$  points in space and measurements are taken across discrete time-periods  $t = 1, \dots, T$  for  $\mathbf{Y}_{s,t}$ . The dependent variable is also a function of design matrix  $\mathbf{X}$  (which may also be dynamic, but is static in this example) and dynamic regression effects matrix  $\beta_{1:J,1:T}$ . For more information on kriging, see section 34. For more information on state-space or a DLM, see section 24. To extend this to a large spatial data set, consider incorporating the predictive process kriging example in section 35.

### 61.1. Form

$$\mathbf{Y}_{s,t} \sim \mathcal{N}(\mu_{s,t}, \sigma_1^2), \quad s = 1, \dots, S, \quad t = 1, \dots, T$$

$$\mu_{s,t} = \mathbf{X}_{s,1:J} \beta_{1:J,t} + \Theta_{s,t}$$

$$\begin{aligned}
\Theta_{s,t} &= \frac{\Sigma_{s,s,t}}{\sum_{r=1}^S \Sigma_{r,s,t}} \Theta_{s,t-1}, \quad s = 1, \dots, S, \quad t = 2, \dots, T \\
\Theta_{s,1} &= \zeta_s \\
\zeta &\sim \mathcal{N}_S(0, \Sigma_{1:S,1:S,1}) \\
\Sigma_{1:S,1:S,t} &= \lambda_t^2 \exp(-\phi_t \mathbf{D})^{\kappa[t]} \\
\sigma_1 &\sim \mathcal{HC}(25) \\
\beta_{j,1} &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 2 \\
\beta_{1,t} &\sim \mathcal{N}(\beta_{1,t-1}, \sigma_2^2), \quad t = 2, \dots, T \\
\beta_{2,t} &\sim \mathcal{N}(\beta_{2,t-1}, \sigma_3^2), \quad t = 2, \dots, T \\
\phi_1 &\sim \mathcal{HN}(1000) \\
\phi_t &\sim \mathcal{N}(\phi_{t-1}, \sigma_4^2) \in [0, \infty], \quad t = 2, \dots, T \\
\kappa_1 &\sim \mathcal{HN}(1000) \\
\kappa_t &\sim \mathcal{N}(\kappa_{t-1}, \sigma_5^2) \in [0, \infty], \quad t = 2, \dots, T \\
\lambda_1 &\sim \mathcal{HN}(1000) \\
\lambda_t &\sim \mathcal{N}(\lambda_{t-1}, \sigma_6^2) \in [0, \infty], \quad t = 2, \dots, T
\end{aligned}$$

## 61.2. Data

```

S <- 20
T <- 10
longitude <- runif(S,0,100)
latitude <- runif(S,0,100)
D <- as.matrix(dist(cbind(longitude,latitude), diag=TRUE, upper=TRUE))
beta <- matrix(c(50,2), 2, T)
phi <- rep(1,T); kappa <- rep(1.5,T); lambda <- rep(10000,T)
for (t in 2:T) {
  beta[1,t-1] <- beta[1,t-1] + rnorm(1,0,1)
  beta[2,t-1] <- beta[2,t-1] + rnorm(1,0,0.1)
  phi[t] <- phi[t-1] + rnorm(1,0,0.1)
  if(phi[t] < 0.001) phi[t] <- 0.001
  kappa[t] <- kappa[t-1] + rnorm(1,0,0.1)
  lambda[t] <- lambda[t-1] + rnorm(1,0,1000)}
Sigma <- array(0, dim=c(S,S,T))
for (t in 1:T) {
  Sigma[, ,t] <- lambda[t] * exp(-phi[t] * D)^kappa[t]}
zeta <- as.vector(apply(rmvn(1000, rep(0,S), Sigma[, ,1]), 2, mean))
mu <- Theta <- matrix(zeta,S,T)
for (t in 2:T) {for (s in 1:S) {
  Theta[,t] <- sum(Sigma[,s,t] / sum(Sigma[,s,t]) * Theta[,t-1])}}
X <- matrix(runif(S*2,-2,2),S,2); X[,1] <- 1
for (t in 1:T) {mu[,t] <- as.vector(tcrossprod(beta[,t], X))}

```

```

Y <- mu + Theta + matrix(rnorm(S*T,0,0.1),S,T)
mon.names <- c("LP", parm.names(list(sigma=rep(0,6))))
parm.names <- parm.names(list(zeta=rep(0,S), beta=matrix(0,2,T),
  log.phi=rep(0,T), log.kappa=rep(0,T), log.lambda=rep(0,T),
  log.sigma=rep(0,6)))
MyData <- list(D=D, S=S, T=T, X=X, Y=Y, latitude=latitude, longitude=longitude,
  mon.names=mon.names, parm.names=parm.names)

```

### 61.3. Initial Values

```

Initial.Values <- c(rep(0,S), rep(c(mean(Y),0),T), log(rep(1,T)),
  log(rep(1,T)), rep(1,T), log(rep(1,6)))

```

### 61.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  beta <- matrix(parm[grep("beta", Data$parm.names)], 2, Data$T)
  zeta <- parm[grep("zeta", Data$parm.names)]
  phi <- exp(parm[grep("log.phi", Data$parm.names)])
  kappa <- exp(parm[grep("log.kappa", Data$parm.names)])
  lambda <- exp(parm[grep("log.lambda", Data$parm.names)])
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  Sigma <- array(0, dim=c(Data$S, Data$S, Data$T))
  for (t in 1:Data$T) {
    Sigma[ , ,t] <- lambda[t]^2 * exp(-phi[t] * Data$D)^kappa[t]}
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta[,1], 0, sqrt(1000), log=TRUE),
    dnorm(beta[,-1], beta[,-Data$T], matrix(sigma[2:3], 2,
    Data$T-1), log=TRUE))
  zeta.prior <- dmvn(zeta, rep(0,Data$S), Sigma[ , , 1], log=TRUE)
  phi.prior <- sum(dhalfnorm(phi[1], sqrt(1000), log=TRUE),
    dtrunc(phi[-1], "norm", a=0, b=Inf, mean=phi[-Data$T],
    sd=sigma[4], log=TRUE))
  kappa.prior <- sum(dhalfnorm(kappa[1], sqrt(1000), log=TRUE),
    dtrunc(kappa[-1], "norm", a=0, b=Inf, mean=kappa[-Data$T],
    sd=sigma[5], log=TRUE))
  lambda.prior <- sum(dhalfnorm(lambda[1], sqrt(1000), log=TRUE),
    dtrunc(lambda[-1], "norm", a=0, b=Inf, mean=lambda[-Data$T],
    sd=sigma[6], log=TRUE))
  sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
  ### Log-Likelihood
  mu <- Theta <- matrix(zeta, Data$S, Data$T)
  mu[,1] <- as.vector(tcrossprod(beta[,1], Data$X))
  for (t in 2:Data$T) {

```

```

    mu[,t] <- as.vector(tcrossprod(beta[,t], Data$X))
    for (s in 1:Data$S) {
      Theta[,t] <- Sigma[,s,t] / sum(Sigma[,s,t]) * Theta[,t-1]}
  mu <- mu + Theta
  LL <- sum(dnorm(Data$Y, mu, sigma[1], log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + zeta.prior + sum(phi.prior) +
    sum(kappa.prior) + sum(lambda.prior) + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma),
    yhat=mu, parm=parm)
  return(Modelout)
}

```

## 62. Space-Time, Nonseparable

This approach to space-time or spatiotemporal modeling applies kriging both to the stationary spatial and temporal components, where space is continuous and time is discrete. Matrix  $\Xi$  contains the space-time effects. Spatial coordinates are given in longitude and latitude for  $s = 1, \dots, S$  points in space and measurements are taken across time-periods  $t = 1, \dots, T$  for  $\mathbf{Y}_{s,t}$ . The dependent variable is also a function of design matrix  $\mathbf{X}$  and regression effects vector  $\beta$ . For more information on kriging, see section 34. This example uses a nonseparable, stationary covariance function in which space and time are separable only when  $\psi = 0$ . To extend this to a large space-time data set, consider incorporating the predictive process kriging example in section 35.

### 62.1. Form

$$\mathbf{Y}_{s,t} \sim \mathcal{N}(\mu_{s,t}, \sigma_1^2), \quad s = 1, \dots, S, \quad t = 1, \dots, T$$

$$\mu = \mathbf{X}\beta + \Xi$$

$$\Xi \sim \mathcal{N}_{ST}(\Xi_\mu, \Sigma)$$

$$\Sigma = \sigma_2^2 \exp \left( -\frac{\mathbf{D}_S^\kappa}{\phi_1} - \frac{\mathbf{D}_T^\lambda}{\phi_2} - \psi \frac{\mathbf{D}_S^\kappa}{\phi_1} \frac{\mathbf{D}_T^\lambda}{\phi_2} \right)$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

$$\phi_k \sim \mathcal{U}(1, 5), \quad k = 1, \dots, 2$$

$$\sigma_k \sim \mathcal{HC}(25), \quad k = 1, \dots, 2$$

$$\psi \sim \mathcal{HC}(25)$$

$$\Xi_\mu = 0$$

$$\kappa = 1, \quad \lambda = 1$$

## 62.2. Data

```

S <- 10
T <- 5
longitude <- runif(S,0,100)
latitude <- runif(S,0,100)
D.S <- as.matrix(dist(cbind(rep(longitude,T),rep(latitude,T)), diag=TRUE,
  upper=TRUE))
D.T <- as.matrix(dist(cbind(rep(1:T,each=S),rep(1:T,each=S)), diag=TRUE,
  upper=TRUE))
Sigma <- 10000 * exp(-D.S/3 - D.T/2 - 0.2*(D.S/3)*(D.T/2))
Xi <- as.vector(apply(rmvn(1000, rep(0,S*T), Sigma), 2, mean))
Xi <- matrix(Xi,S,T)
beta <- c(50,2)
X <- matrix(runif(S*2,-2,2),S,2); X[,1] <- 1
mu <- as.vector(tcrossprod(beta, X))
Y <- mu + Xi
mon.names <- c("LP","psi","sigma[1]","sigma[2]")
parm.names <- parm.names(list(Xi=matrix(0,S,T), beta=rep(0,2),
  phi=rep(0,2), log.sigma=rep(0,2), log.psi=0))
MyData <- list(D.S=D.S, D.T=D.T, S=S, T=T, X=X, Y=Y, latitude=latitude,
  longitude=longitude, mon.names=mon.names, parm.names=parm.names)

```

## 62.3. Initial Values

```
Initial.Values <- c(rep(0,S*T), mean(Y), 0, rep(1,2), rep(0,2), 0)
```

## 62.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  Xi.mu <- rep(0,Data$S*Data$T)
  ### Parameters
  beta <- parm[grepl("beta", Data$parm.names)]
  Xi <- parm[grepl("Xi", Data$parm.names)]
  kappa <- 1; lambda <- 1
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  phi <- interval(parm[grepl("phi", Data$parm.names)], 1, 5)
  parm[grepl("phi", Data$parm.names)] <- phi
  psi <- exp(parm[grepl("log.psi", Data$parm.names)])
  Sigma <- sigma[2]*sigma[2] * exp(-(Data$D.S / phi[1])^kappa -
    (Data$D.T / phi[2])^lambda -
    psi*(Data$D.S / phi[1])^kappa * (Data$D.T / phi[2])^lambda)
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  Xi.prior <- dmvn(Xi, Xi.mu, Sigma, log=TRUE)
}

```

```

sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
phi.prior <- sum(dunif(phi, 1, 5, log=TRUE))
psi.prior <- dhalfcauchy(psi, 25, log=TRUE)
### Log-Likelihood
Xi <- matrix(Xi, Data$S, Data$T)
mu <- as.vector(tcrossprod(beta, Data$X)) + Xi
LL <- sum(dnorm(Data$Y, mu, sigma[1], log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + Xi.prior + sigma.prior + phi.prior + psi.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,psi,sigma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

### 63. Space-Time, Separable

This introductory approach to space-time or spatiotemporal modeling applies kriging both to the stationary spatial and temporal components, where space is continuous and time is discrete. Vector  $\zeta$  contains the spatial effects and vector  $\theta$  contains the temporal effects. Spatial coordinates are given in longitude and latitude for  $s = 1, \dots, S$  points in space and measurements are taken across time-periods  $t = 1, \dots, T$  for  $\mathbf{Y}_{s,t}$ . The dependent variable is also a function of design matrix  $\mathbf{X}$  and regression effects vector  $\beta$ . For more information on kriging, see section 34. This example uses separable space-time covariances, which is more convenient but usually less appropriate than a nonseparable covariance function. To extend this to a large space-time data set, consider incorporating the predictive process kriging example in section 35.

#### 63.1. Form

$$\begin{aligned}
\mathbf{Y}_{s,t} &\sim \mathcal{N}(\mu_{s,t}, \sigma_1^2), \quad s = 1, \dots, S, \quad t = 1, \dots, T \\
\mu_{s,t} &= \mathbf{X}_{s,1:T}\beta + \zeta_s + \Theta_{s,t} \\
\Theta_{s,1:T} &= \theta \\
\theta &\sim \mathcal{N}_N(\theta_\mu, \Sigma_T) \\
\Sigma_T &= \sigma_3^2 \exp(-\phi_2 \mathbf{D}_T)^\lambda \\
\zeta &\sim \mathcal{N}_N(\zeta_\mu, \Sigma_S) \\
\Sigma_S &= \sigma_2^2 \exp(-\phi_1 \mathbf{D}_S)^\kappa \\
\beta_j &\sim \mathcal{N}(0, 1000), \quad j = 1, \dots, 2 \\
\sigma_k &\sim \mathcal{HC}(25), \quad k = 1, \dots, 3 \\
\phi_k &\sim \mathcal{U}(1, 5), \quad k = 1, \dots, 2 \\
\zeta_\mu &= 0
\end{aligned}$$

$$\theta_\mu = 0$$

$$\kappa = 1, \quad \lambda = 1$$

### 63.2. Data

```

S <- 20
T <- 10
longitude <- runif(S,0,100)
latitude <- runif(S,0,100)
D.S <- as.matrix(dist(cbind(longitude,latitude), diag=TRUE, upper=TRUE))
Sigma.S <- 10000 * exp(-1.5 * D.S)
zeta <- as.vector(apply(rmvn(1000, rep(0,S), Sigma.S), 2, mean))
D.T <- as.matrix(dist(cbind(c(1:T),c(1:T)), diag=TRUE, upper=TRUE))
Sigma.T <- 10000 * exp(-3 * D.T)
theta <- as.vector(apply(rmvn(1000, rep(0,T), Sigma.T), 2, mean))
Theta <- matrix(theta,S,T,byrow=TRUE)
beta <- c(50,2)
X <- matrix(runif(S*2,-2,2),S,2); X[,1] <- 1
mu <- as.vector(tcrossprod(beta, X))
Y <- mu + zeta + Theta + matrix(rnorm(S*T,0,0.1),S,T)
mon.names <- c("LP","sigma[1]","sigma[2]","sigma[3]")
parm.names <- parm.names(list(zeta=rep(0,S), theta=rep(0,T),
  beta=rep(0,2), phi=rep(0,2), log.sigma=rep(0,3)))
MyData <- list(D.S=D.S, D.T=D.T, S=S, T=T, X=X, Y=Y, latitude=latitude,
  longitude=longitude, mon.names=mon.names, parm.names=parm.names)

```

### 63.3. Initial Values

```

Initial.Values <- c(rep(0,S), rep(0,T), rep(0,2), rep(1,2), rep(0,3))

```

### 63.4. Model

```

Model <- function(parm, Data)
{
  ### Hyperparameters
  zeta.mu <- rep(0,Data$S)
  theta.mu <- rep(0,Data$T)
  ### Parameters
  beta <- parm[grep("beta", Data$parm.names)]
  zeta <- parm[grep("zeta", Data$parm.names)]
  theta <- parm[grep("theta", Data$parm.names)]
  kappa <- 1; lambda <- 1
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  phi <- interval(parm[grep("phi", Data$parm.names)], 1, 5)
  parm[grep("phi", Data$parm.names)] <- phi
  Sigma.S <- sigma[2]^2 * exp(-phi[1] * Data$D.S)^kappa

```

```

Sigma.T <- sigma[3]^2 * exp(-phi[2] * Data$D.T)^lambda
### Log(Prior Densities)
beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
zeta.prior <- dmvn(zeta, zeta.mu, Sigma.S, log=TRUE)
theta.prior <- dmvn(theta, theta.mu, Sigma.T, log=TRUE)
sigma.prior <- sum(dhalfcauchy(25, log=TRUE))
phi.prior <- sum(dunif(phi, 1, 5, log=TRUE))
### Log-Likelihood
Theta <- matrix(theta, Data$S, Data$T, byrow=TRUE)
mu <- as.vector(tcrossprod(beta, Data$X)) + zeta + Theta
LL <- sum(dnorm(Data$Y, mu, sigma[1], log=TRUE))
### Log-Posterior
LP <- LL + beta.prior + zeta.prior + theta.prior + sigma.prior +
  phi.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma),
  yhat=mu, parm=parm)
return(Modelout)
}

```

## 64. Threshold Autoregression (TAR)

### 64.1. Form

$$\begin{aligned}
\mathbf{y}_t &\sim \mathcal{N}(\nu_t, \sigma^2), \quad t = 1, \dots, T \\
\mathbf{y}^{new} &= \alpha_2 + \phi_2 \mathbf{y}_T \\
\nu_t &= \begin{cases} \alpha_1 + \phi_1 \mathbf{y}_{t-1}, & t = 1, \dots, T \quad \text{if } t \geq \theta \\ \alpha_2 + \phi_2 \mathbf{y}_{t-1}, & t = 1, \dots, T \quad \text{if } t < \theta \end{cases} \\
\alpha_j &\sim \mathcal{N}(0, 1000) \in [-1, 1], \quad j = 1, \dots, 2 \\
\phi_j &\sim \mathcal{N}(0, 1000) \in [-1, 1], \quad j = 1, \dots, 2 \\
\theta &\sim \mathcal{U}(2, T-1) \\
\sigma &\sim \mathcal{HC}(25)
\end{aligned}$$

### 64.2. Data

```

y <- c(0.02, -0.51, -0.30, 1.46, -1.26, -2.15, -0.91, -0.53, -1.91,
  2.64, 1.64, 0.15, 1.46, 1.61, 1.96, -2.67, -0.19, -3.28,
  1.89, 0.91, -0.71, 0.74, -0.10, 3.20, -0.80, -5.25, 1.03,
  -0.40, -1.62, -0.80, 0.77, 0.17, -1.39, -1.28, 0.48, -1.02,
  0.09, -1.09, 0.86, 0.36, 1.51, -0.02, 0.47, 0.62, -1.36,
  1.12, 0.42, -4.39, -0.87, 0.05, -5.41, -7.38, -1.01, -1.70,
  0.64, 1.16, 0.87, 0.28, -1.69, -0.29, 0.13, -0.65, 0.83,
  0.62, 0.05, -0.14, 0.01, -0.36, -0.32, -0.80, -0.06, 0.24,

```

```

0.23, -0.37, 0.00, -0.33, 0.21, -0.10, -0.10, -0.01, -0.40,
-0.35, 0.48, -0.28, 0.08, 0.28, 0.23, 0.27, -0.35, -0.19,
0.24, 0.17, -0.02, -0.23, 0.03, 0.02, -0.17, 0.04, -0.39,
-0.12, 0.16, 0.17, 0.00, 0.18, 0.06, -0.36, 0.22, 0.14,
-0.17, 0.10, -0.01, 0.00, -0.18, -0.02, 0.07, -0.06, 0.06,
-0.05, -0.08, -0.07, 0.01, -0.06, 0.01, 0.01, -0.02, 0.01,
0.01, 0.12, -0.03, 0.08, -0.10, 0.01, -0.03, -0.08, 0.04,
-0.09, -0.08, 0.01, -0.05, 0.08, -0.14, 0.06, -0.11, 0.09,
0.06, -0.12, -0.01, -0.05, -0.15, -0.05, -0.03, 0.04, 0.00,
-0.12, 0.04, -0.06, -0.05, -0.07, -0.05, -0.14, -0.05, -0.01,
-0.12, 0.05, 0.06, -0.10, 0.00, 0.01, 0.00, -0.08, 0.00,
0.00, 0.07, -0.01, 0.00, 0.09, 0.33, 0.13, 0.42, 0.24,
-0.36, 0.22, -0.09, -0.19, -0.10, -0.08, -0.07, 0.05, 0.07,
0.07, 0.00, -0.04, -0.05, 0.03, 0.08, 0.26, 0.10, 0.08,
0.09, -0.07, -0.33, 0.17, -0.03, 0.07, -0.04, -0.06, -0.06,
0.07, -0.03, 0.00, 0.08, 0.27, 0.11, 0.11, 0.06, -0.11,
-0.09, -0.21, 0.24, -0.12, 0.11, -0.02, -0.03, 0.02, -0.10,
0.00, -0.04, 0.01, 0.02, -0.03, -0.10, -0.09, 0.17, 0.07,
-0.05, -0.01, -0.05, 0.01, 0.00, -0.08, -0.05, -0.08, 0.07,
0.06, -0.14, 0.02, 0.01, 0.04, 0.00, -0.13, -0.17)

```

```

T <- length(y)
mon.names <- c("LP", "sigma", "ynew")
parm.names <- parm.names(list(alpha=rep(0,2), phi=rep(0,2), theta=0,
  log.sigma=0))
MyData <- list(T=T, mon.names=mon.names, parm.names=parm.names, y=y)

```

### 64.3. Initial Values

```
Initial.Values <- c(rep(0,4), T/2, log(1))
```

### 64.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- interval(parm[1:2], -1, 1); parm[1:2] <- alpha
  phi <- interval(parm[3:4], -1, 1); parm[3:4] <- phi
  theta <- interval(parm[5], 2, Data$T-1); parm[5] <- theta
  sigma <- exp(parm[6])
  ### Log(Prior Densities)
  alpha.prior <- sum(dtrunc(alpha, "norm", a=-1, b=1, mean=0,
    sd=sqrt(1000), log=TRUE))
  phi.prior <- sum(dtrunc(phi, "norm", a=-1, b=1, mean=0,
    sd=sqrt(1000), log=TRUE))
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  phi.prior <- sum(dnorm(phi, 0, sqrt(1000), log=TRUE))
}

```

```

theta.prior <- dunif(theta, 2, Data$T-1, log=TRUE)
sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
### Log-Likelihood
mu <- matrix(0, Data$T, 2)
mu[,1] <- c(alpha[1], alpha[1] + phi[1]*Data$y[-Data$T])
mu[,2] <- c(alpha[2], alpha[2] + phi[2]*Data$y[-Data$T])
nu <- ifelse(1:Data$T < theta, mu[,1], mu[,2])
ynew <- alpha[2] + phi[2]*Data$y[Data$T]
LL <- sum(dnorm(Data$y, nu, sigma, log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + phi.prior + theta.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma,ynew),
  yhat=nu, parm=parm)
return(Modelout)
}

```

## 65. Variable Selection

This example uses a modified form of the random-effects (or global adaptation) Stochastic Search Variable Selection (SSVS) algorithm presented in [O'Hara and Sillanpaa \(2009\)](#), which selects variables according to practical significance rather than statistical significance. Here, SSVS is applied to linear regression, though this method is widely applicable. For  $J$  variables, each regression effects vector  $\beta_j$  is conditional on  $\gamma_j$ , a binary inclusion variable. Each  $\beta_j$  is a discrete mixture distribution with respect to  $\gamma_j = 0$  or  $\gamma_j = 1$ , with precision 100 or  $\beta_\sigma = 0.1$ , respectively. As with other representations of SSVS, these precisions may require tuning.

With other representations of SSVS, each  $\gamma_j$  is Bernoulli-distributed, though this would be problematic in Laplace's Demon, because  $\gamma_j$  would be in the list of parameters (rather than monitors), and would not be stationary due to switching behavior. To keep  $\gamma$  in the monitors, an uninformative normal density is placed on each prior  $\delta_j$ , with mean  $1/J$  for  $J$  variables and variance 1000. Each  $\delta_j$  is transformed with the inverse logit and rounded to  $\gamma_j$ . Note that  $\lfloor x + 0.5 \rfloor$  means to round  $x$ . The prior for  $\delta$  can be manipulated to influence sparseness.

When the goal is to select the best model, each  $\mathbf{X}_{1:N,j}$  is retained for a future run when the posterior mean of  $\gamma_j \geq 0.5$ . When the goal is model-averaging, the results of this model may be used directly.

### 65.1. Form

$$\begin{aligned}
\mathbf{y} &\sim \mathcal{N}(\mu, \sigma^2) \\
\mu &= \mathbf{X}\beta \\
(\beta_j | \gamma_j) &\sim (1 - \gamma_j)\mathcal{N}(0, 0.01) + \gamma_j\mathcal{N}(0, \beta_\sigma^2) \quad j = 1, \dots, J \\
\beta_\sigma &\sim \mathcal{HC}(25) \\
\gamma_j &= \lfloor \frac{1}{1 + \exp(-\delta_j)} + 0.5 \rfloor, \quad j = 1, \dots, J
\end{aligned}$$

$$\delta_j \sim \mathcal{N}(0, 10) \in [-100, 100], \quad j = 1, \dots, J$$

$$\sigma \sim \mathcal{HC}(25)$$

## 65.2. Data

```
data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
mon.names <- c("LP", "min.beta.sigma", "sigma",
  parm.names(list(gamma=rep(0,J)))
parm.names <- parm.names(list(beta=rep(0,J), delta=rep(0,J),
  log.beta.sigma=0, log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, y=y)
```

## 65.3. Initial Values

```
Initial.Values <- c(rep(0,J), rep(0,J), log(1), log(1))
```

## 65.4. Model

```
Model <- function(parm, Data)
{
  ### Hyperparameters
  beta.sigma <- exp(parm[grepl("log.beta.sigma", Data$parm.names)])
  delta <- interval(parm[grepl("delta", Data$parm.names)], -100, 100)
  parm[grepl("delta", Data$parm.names)] <- delta
  ### Parameters
  beta <- parm[1:Data$J]
  gamma <- round(invlogit(delta))
  beta.sigma <- ifelse(gamma == 0, 0.1, beta.sigma)
  sigma <- exp(parm[grepl("log.sigma", Data$parm.names)])
  ### Log(Hyperprior and Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, beta.sigma, log=TRUE))
  beta.sigma.prior <- sum(dhalfcauchy(beta.sigma, 25, log=TRUE))
  delta.prior <- sum(dtrunc(delta, "norm", a=-100, b=100,
    mean=logit(1/Data$J), sd=sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(dnorm(y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + beta.sigma.prior + delta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP, min(beta.sigma),
    sigma, gamma), yhat=mu, parm=parm)
```

```
return(Modelout)
}
```

## 66. Vector Autoregression, VAR(1)

### 66.1. Form

$$\begin{aligned}
\mathbf{Y}_{t,j} &\sim \mathcal{N}(\mu_{t,j}, \sigma_j^2), \quad t = 1, \dots, T, \quad j = 1, \dots, J \\
\mu_{t,j} &= \alpha_j + \Phi_{1:J,j} \mathbf{Y}_{t-1,j} \\
\mathbf{y}_j^{new} &= \alpha_j + \Phi_{1:J,j} \mathbf{Y}_{T,j} \\
\alpha_j &\sim \mathcal{N}(0, 1000) \\
\sigma_j &\sim \mathcal{HC}(25) \\
\Phi_{i,k} &\sim \mathcal{N}(0, 1000), \quad i = 1, \dots, J, \quad k = 1, \dots, J
\end{aligned}$$

### 66.2. Data

```
T <- 100
J <- 3
Y <- matrix(0,T,J)
for (j in 1:J) {for (t in 2:T) {
  Y[t,j] <- Y[t-1,j] + rnorm(1,0,0.1)}}
mon.names <- c("LP", parm.names(list(ynew=rep(0,J))))
parm.names <- parm.names(list(alpha=rep(0,J), Phi=matrix(0,J,J),
  log.sigma=rep(0,J)))
MyData <- list(J=J, T=T, Y=Y, mon.names=mon.names, parm.names=parm.names)
```

### 66.3. Initial Values

```
Initial.Values <- c(colMeans(Y), rep(0,J*J), rep(log(1),J))
```

### 66.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1:Data$J]
  Phi <- matrix(parm[grep("Phi", Data$parm.names)], Data$J, Data$J)
  sigma <- exp(parm[grep("log.sigma", Data$parm.names)])
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  Phi.prior <- sum(dnorm(Phi, 0, sqrt(1000), log=TRUE))
}
```

```

sigma.prior <- sum(dhalfcauchy(sigma, 25, log=TRUE))
### Log-Likelihood
mu <- matrix(alpha,Data$T,Data$J,byrow=TRUE)
  mu[-1,] <- mu[-1,] + t(tcrossprod(Phi,Data$Y[-Data$T,]))
ynew <- alpha + as.vector(crossprod(Phi, Data$Y[Data$T,]))
LL <- sum(dnorm(Data$Y, mu,
  matrix(sigma,Data$T,Data$J,byrow=TRUE), log=TRUE))
### Log-Posterior
LP <- LL + alpha.prior + Phi.prior + sigma.prior
Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,ynew), yhat=mu,
  parm=parm)
return(Modelout)
}

```

## 67. Weighted Regression

It is easy enough to apply record-level weights to the likelihood. Here, weights are applied to the linear regression example in section 37.

### 67.1. Form

$$\mathbf{y} \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = \mathbf{X}\beta$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J$$

$$\sigma \sim \mathcal{HC}(25)$$

### 67.2. Data

```

data(demonsnacks)
N <- NROW(demonsnacks)
J <- NCOL(demonsnacks)
y <- log(demonsnacks$Calories)
X <- cbind(1, as.matrix(demonsnacks[,c(1,3:10)]))
for (j in 2:J) {X[,j] <- CenterScale(X[,j])}
w <- c(rep(1,5), 0.2, 1, 0.01, rep(1,31))
w <- w * (sum(w) / N)
mon.names <- c("LP","sigma")
parm.names <- parm.names(list(beta=rep(0,J), log.sigma=0))
MyData <- list(J=J, X=X, mon.names=mon.names, parm.names=parm.names, w=w,
  y=y)

```

### 67.3. Initial Values

```
Initial.Values <- c(rep(0,J), log(1))
```

### 67.4. Model

```
Model <- function(parm, Data)
{
  ### Parameters
  beta <- parm[1:Data$J]
  sigma <- exp(parm[Data$J+1])
  ### Log(Prior Densities)
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  sigma.prior <- dhalfcauchy(sigma, 25, log=TRUE)
  ### Log-Likelihood
  mu <- tcrossprod(beta, Data$X)
  LL <- sum(w * dnorm(Data$y, mu, sigma, log=TRUE))
  ### Log-Posterior
  LP <- LL + beta.prior + sigma.prior
  Modelout <- list(LP=LP, Dev=-2*LL, Monitor=c(LP,sigma), yhat=mu,
    parm=parm)
  return(Modelout)
}
```

## 68. Zero-Inflated Poisson (ZIP)

### 68.1. Form

$$\mathbf{y} \sim \mathcal{P}(\Lambda_{1:N,2})$$

$$\mathbf{z} \sim \mathcal{BERN}(\Lambda_{1:N,1})$$

$$\mathbf{z}_i = \begin{cases} 1 & \text{if } \mathbf{y}_i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda_{i,2} = \begin{cases} 0 & \text{if } \Lambda_{i,1} \geq 0.5 \\ \Lambda_{i,2} & \text{otherwise} \end{cases}$$

$$\Lambda_{1:N,1} = \frac{1}{1 + \exp(-\mathbf{X}_1 \alpha)}$$

$$\Lambda_{1:N,2} = \exp(\mathbf{X}_2 \beta)$$

$$\alpha_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J_1$$

$$\beta_j \sim \mathcal{N}(0, 1000), \quad j = 1, \dots, J_2$$

## 68.2. Data

```

N <- 1000
J1 <- 4
J2 <- 3
X1 <- matrix(runif(N*J1,-2,2),N,J1); X1[,1] <- 1
X2 <- matrix(runif(N*J2,-2,2),N,J2); X2[,1] <- 1
alpha <- runif(J1,-1,1)
beta <- runif(J2,-1,1)
p <- as.vector(invlogit(tcrossprod(alpha, X1) + rnorm(N,0,0.1)))
mu <- as.vector(round(exp(tcrossprod(beta, X2) + rnorm(N,0,0.1))))
y <- ifelse(p > 0.5, 0, mu)
z <- ifelse(y == 0, 1, 0)
mon.names <- "LP"
parm.names <- parm.names(list(alpha=rep(0,J1), beta=rep(0,J2)))
MyData <- list(J1=J1, J2=J2, N=N, X1=X1, X2=X2, mon.names=mon.names,
               parm.names=parm.names, y=y, z=z)

```

## 68.3. Initial Values

```

Initial.Values <- rep(0,J1+J2)

```

## 68.4. Model

```

Model <- function(parm, Data)
{
  ### Parameters
  alpha <- parm[1:Data$J1]
  beta <- parm[grep("beta", Data$parm.names)]
  ### Log(Prior Densities)
  alpha.prior <- sum(dnorm(alpha, 0, sqrt(1000), log=TRUE))
  beta.prior <- sum(dnorm(beta, 0, sqrt(1000), log=TRUE))
  ### Log-Likelihood
  Lambda <- matrix(NA, Data$N, 2)
  Lambda[,1] <- invlogit(tcrossprod(alpha, Data$X1))
  Lambda[,2] <- exp(tcrossprod(beta, Data$X2))
  Lambda[,2] <- ifelse(Lambda[,1] >= 0.5, 0, Lambda[,2])
  LL1 <- sum(dbern(Data$z, Lambda[,1], log=TRUE))
  LL2 <- sum(dpois(Data$y, Lambda[,2], log=TRUE))
  ### Log-Posterior
  LP <- LL1 + LL2 + alpha.prior + beta.prior
  Modelout <- list(LP=LP, Dev=-2*LL2, Monitor=LP,
                  yhat=Lambda[,2], parm=parm)
  return(Modelout)
}

```

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